Math 310 **Final, Fall 2008, Due 5pm, Dec. 18**

You can quote all the theorems proven in class, homework, midterm problems and online class notes to solve the problems below.

(1) Show that if $a_n \geq b_n$ for all $n \in \mathbb{N}$ and if $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then $a \geq b$. (10 points)

(2a) Show that for any two *real* numbers x and y with $x < y$, there is a *rational* numbers r such that $x < r < y$. Conclude that therefore there are two rational numbers r and s such that $x < r < s < y$. (10 points).

(Hint: midterm problem 5.)

(2b) We have shown in class that there are uncountably many real numbers in the open interval $(0, 1)$. Show that there are uncountably many *irrational numbers* in the interval $(0, 1)$. (5 points)

(2c) For any two *rational* numbers r and s with $r < s$, show that there are uncountably many *irrational* numbers in the open interval (r, s). (10 points)

(Hint: Scaling $(0, 1)$ appropriately.)

(2d) Show that for any two real numbers x and y with $x < y$, there are uncountably many *irrational* numbers in the interval (x, y). (5 points)

(3a) Use the pigeonhole principle (midterm problem 4(b)) to prove that a rational number r with $0 < r < 1$ is either a finite decimal of the form

$$0.a_1a_2a_3 \cdots a_k$$

for some k, or an infinite periodical decimal of the form

$$0.a_1a_2 \cdots a_s b_1 b_2 \cdots b_t b_1 b_2 \cdots b_t b_1 b_2 \cdots b_t \cdots,$$

where the b-digits are periodic. (10 points)

(3b) Conversely, a finite decimal or an infinite periodical decimal is a rational number. Do not give the general proof. You are only asked to express the periodical

$$0.034123123123123 \cdots,$$

where the digits 123 repeat, as a fraction q/p in lowest terms, where q and p are natural numbers. (5 points)

(4) Let $a_1 = \sqrt{2}, a_2 = \sqrt{2 + \sqrt{2}}, a_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \ldots$. In general, $a_{n+1} = \sqrt{2 + a_n}$.

(a) Show that $\lim_{n\to\infty} a_n$ exists. (10 points)

(Hint: Show that the sequence is increasing and bounded.)

(b) Find the exact limit of the sequence (a_n). (5 points)

(5) Find the smallest natural number x satisfying

$$\begin{aligned} 2x &\equiv 5 \mod 7 \\ 4x &\equiv 2 \mod 6 \\ 3x &\equiv 4 \mod 5. \end{aligned}$$

(10 points)

(Hint: Convert the equations so that the coefficients for x are all 1.)