Math 310 Midterm, Fall 2008, Due Nov. 7

(1a) Use the mathematical induction to show that for all $n \in \mathbb{N}$,

$$1 + 2 + 3 + \cdots + n = n(n+1)/2. \qquad \text{(5 points)}$$

(1b) Use $(n+1)^3 - n^3 = 3n^2 + 3n + 1$ and (1a) to show

$$1 + 2^2 + 3^2 + \cdots n^2 = n(n+1)(2n+1)/6 \qquad \text{(5 points)}$$

(1c) Use the method in (1b) to find

$$1 + 2^3 + 3^3 + \cdots n^3. \qquad \text{(5 points)}$$

(Renark: Of course, (1a) can also be done by the method of (1b) by quoting the identity $(n+1)^2 - n^2 = 2n+1$. However, you must use the mathematical induction to solve (1a).)

(2a) Use the Euclidean division algorithm to show that for any two natural numbers m and n, there is a natural number q such that $mq > n$. (10 points)

(Hint: Divide n by m.)

(2b) Conclude that for any two positive rational numbers r and s, there is a natural number n such that $nr > s$. (5 points)

(Remark: (2b) is the reason why you can pick the first natural number $N \geq$ a given positive rational number, say, $10/\epsilon$, in all of our $\epsilon - N$ proofs.)

(3) Show that the definition of the product of two real numbers is well-defined independent of the representatives chosen. (10 points)

(4a) Let $I_n = \{k \in \mathbb{N} : k \leq n\}$. Suppose $f : I_n \to \mathbb{N}$ is a one-to-one function, that is $f(x) = f(y)$ implies $x = y$. Let $Im(f)$ be the image of f in $\mathbb{N}$. Use the mathematical induction to show that there is another one-to-one function $g : I_n \to Im(f)$ such that g is an onto map, that is, each element in $Im(f)$ is the image of some element in I_n via g, and moreover $g(x) < g(y)$ if $x < y$. (10 points)

(Remark: In spite of its dry statement, the problem simply says we can reshuffle the image elements so that the new map is order-preserving. For instance, if

$$f : 1 \mapsto 10, \quad 2 \mapsto 28, \quad 3 \mapsto 6,$$

then

$$g : 1 \mapsto 6, \quad 2 \mapsto 10, \quad 3 \mapsto 28.$$

Here, $I_n = \{1, 2, 3\}$ and $Im(f) = \{6, 10, 28\}$.

(4b) Conclude that if $m < n$ then a function $f : I_n \to I_m$ *cannot* be one-to-one. (5 points)

(Remark: This is the Pigeon Hole Principle that says that if the number of pigeons is greater than the number of pigeon holes, then there must be a pigeon hole with at least two pigeons.)

(5) Show that between any two real numbers x and y satisfying $x+1 < y$, there is an interger n such that $x < n < y$. (10 points)

(Hint: Without loss of generality, we may assume $0 \leq x \leq y$ (why?). Consider the nonempty set of natural numbers $\geq y$ (why is this set nonempty?). Apply the well-ordering principle to this set.)

(6) Show that $n^n/n!$ is *not* Cauchy. (10 points)