

Math 310 Class Notes 3

We are given the set \mathbb{N} of natural numbers that satisfies the five axioms of Peano.

Peano's Axioms

- (1): $1 \in \mathbb{N}$.
- (2): For any $n \in \mathbb{N}$, there is one and only one natural number, denoted by n' , called the *successor* of n .
- (3): If $m, n \in \mathbb{N}$ such that $m' = n'$, then $m = n$. (No two different natural numbers have the same successor.)
- (4): For any $m \in \mathbb{N}$, we have $m' \neq 1$. (No natural number has 1 as a successor.)
- (5): Let $S \subset \mathbb{N}$ have the following properties:
 - (a): $1 \in S$.
 - (b): $k \in S$ implies $k' \in S$.

Then we can conclude that $S = \mathbb{N}$. (This is Mathematical Induction.)

Next, we introduce the notion of addition, which is a binary operation $+$ assigning any natural numbers m and n to a third natural number, denoted $m + n$.

Remark 1. *Strictly speaking, the word "assign" needs justification. It will be seen later when we introduce the notion of a function that $+$ is actually a function. We leave it slightly vague, though intuitively clear, as is now.*

Definition 2. *For all $n \in \mathbb{N}$, assign*

$$1 + n = n'.$$

Moreover, if $m + n$ is assigned for m and all $n \in \mathbb{N}$, assign

$$m' + n = (m + n)'$$

for m' and all n .

For instance, Since $1 + n = n'$ is assigned, Definition 2 gives

$$2 + n = 1' + n = (1 + n)' = (n)',$$

so that $2 + n$ is assigned for all n . Continuing in this fashion,

$$3 + n = 2' + n = (2 + n)' = ((n'))',$$

so that $3 + n$ is assigned for all n , etc. The following theorem is thus not surprising.

Theorem 3. $+$ in Definition 2 can be extended to one and only one binary operation, also denoted by $+$, satisfying the following two properties:

- (A): $m' = m + 1$.
 (B): $(m + n)' = m + n'$.

Proof. **Existence of $+$.**

Step 1. By Definition 2,

$$(1) \quad 1 + n = n'$$

for all n .

Step 2. Let $S \subset \mathbb{N}$ be the set of all m such that $+$ can be assigned, according to Definition 2, satisfying (A) and (B) for m and all n

Step 3. We show $m = 1 \in S$. That is, from (1) we must establish (A) and (B). Now, setting $n = 1$ in (1), we obtain

$$(2) \quad 1 + 1 = 1'$$

It follows that, with $m = 1$,

$$\begin{aligned} m' &= 1' & (m = 1) \\ &= 1 + 1 & \text{(by (2))} \\ &= m + 1 & (m = 1) \end{aligned}$$

so that (A) holds true for $m = 1$. To establish (B) for $m = 1$, observe that

$$\begin{aligned} (m + n)' &= (1 + n)' & (m = 1) \\ &= (n')' & \text{(by (1))} \\ &= 1 + n' & \text{(by (1), replacing } n \text{ by } n' \text{ in it)} \\ &= m + n' & (m = 1) \end{aligned}$$

Hence (B) holds true for $m = 1$. Therefore, $1 \in S$.

Step 4. We show that $m \in S$ implies $m' \in S$. To this end, observe that $m \in S$ means $m + n$ is already assigned with (A) and (B) true for all n , i.e., with

$$(3) \quad \begin{aligned} m' &= m + 1 \\ (m + n)' &= m + n' \end{aligned}$$

To show that $m' \in S$, observe next that by Definition 2, we have

$$(4) \quad m' + n = (m + n)'$$

for all n . All we need to do to show $m' \in S$ is to ensure that (A) and (B) remain true when we replace m by m' in them. In other words, we want to show

$$(5) \quad \begin{aligned} (m')' &= m' + 1 \\ (m' + n)' &= m' + n' \end{aligned}$$

However,

$$\begin{aligned} m' + 1 &= (m + 1)' && \text{(by (4) with } n = 1) \\ &= (m')' && \text{(by the first equation of (3))} \end{aligned}$$

Hence the first equation of (5) is true. On the other hand,

$$\begin{aligned} (m' + n)' &= ((m + n)')' && \text{(by (4))} \\ &= (m + n')' && \text{(by the second equation of (3))} \\ &= m' + n' && \text{by (4) with } n \text{ replaced by } n' \end{aligned}$$

Hence, the second equation of (5) is verified.

Step 5 Steps 3 and 4 fulfill the fifth axiom of Peano. So, we conclude that $S = \mathbb{N}$.

In summary, we have shown that we can define $+$ for any pair m, n of natural numbers.

Uniqueness of $+$.

Suppose there is another binary operation \oplus such that \oplus is defined for any pair m, n of natural numbers satisfying (A) and (B) in Theorem 3. Fix m . Let S be the set of all n such that

$$m + n = m \oplus n$$

Now $n = 1 \in S$; this is because by (A)

$$m + 1 = m' = m \oplus 1$$

We next show that $n \in S$ implies $n' \in S$. Now, $n \in S$ means

$$(6) \quad m + n = m \oplus n$$

Therefore,

$$\begin{aligned} m + n' &= (m + n)' && \text{(by (B))} \\ &= (m \oplus n)' && \text{(by (6))} \\ &= m \oplus n' && \text{(by (B))} \end{aligned}$$

Therefore, $n' \in S$. Thus, the mathematical induction implies that $S = \mathbb{N}$. In other words, we have shown that

$$m + n = m \oplus n$$

for any pair $m, n \in \mathbb{N}$. That is, there is only one such binary operation.

