ISOPARAMETRIC HYPERSURFACES WITH FOUR PRINCIPAL CURVATURES, IV

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Abstract

We prove that an isoparametric hypersurface with four principal curvatures and multiplicity pair $(7, 8)$ is either the one constructed by Ozeki and Takeuchi, or one of the two constructed by Ferus, Karcher, and Münzner. This completes the classification of isoparametric hypersurfaces in spheres that É. Cartan initiated in the late 1930s.

1. Introduction

The class of isoparametric hypersurfaces with four principal curvatures and multiplicity pair $(7, 8)$ in S^{31} is the only one that has remained unclassified [**1**], [**3**], [**5**], [**14**], [**25**], [**26**]. The subtlety of a possible classification suggests itself when one looks into the three existing examples that are all inhomogeneous, where the octonion algebra is in full force to interplay with the underlying geometric structure, in contrast to the three other anomalous classes of respective multiplicity pairs $(3, 4), (4, 5)$, and $(6, 9)$, where one category (out of at most two) of each class is homogeneous that carries more manageable structural data for the classification [**1**], [**3**], [**5**].

From an algebraic point of view, a classification must begin with classifying the orthogonal multiplications of type $[7, 8, 15]$, i.e., classifying those bilinear maps

$$F : \mathbb{R}^7 \times \mathbb{R}^8 \to \mathbb{R}^{15}$$

satisfying $|F(x, y)| = |F(x)||F(y)|$, or more conveniently for our setup, classifying the following quadratic composition formula of type $[7, 8, 15]$

$$(x_1^2 + \cdots + x_7^2)(y_1^2 + \cdots + y_8^2) = z_1^2 + \cdots + z_{15}^2,$$

where $z_1, \cdots , z_{15}$ are bilinear in $x_1, \cdots , x_7$ and $y_1, \cdots , y_8$, as can be seen by a glance at the first two identities in (2.3) below. Indeed, the composition formula is equivalent to the Hurwitz matrix equations

$$F_aF_b^{tr} + F_bF_a^{tr} = 2\delta_{ab}I_8, \quad 1 \le a, b \le 7,$$

Mathematics Subject Classification. Primary 53C40.

Key words and phrases. Isoparametric hypersurfaces.

where

$$F_a := \begin{pmatrix} A_a & \sqrt{2}B_a \end{pmatrix}$$

for A_a of size 8-by-8 and B_a of sixe 8-by-7. With F_a in place one next solves the same problem for another set of seven matrices

$$G_a := \begin{pmatrix} A_a^{tr} & \sqrt{2}C_a \end{pmatrix}$$

for some C_a of size 8-by-7. Then A_a, B_a, C_a are candidates to form the shape operator S_a, in the normal a-direction, of the shape operator of the focal manifold M_+ of the isoparametric hypersurface of the smaller codimension ($= 8$) in the sphere S^{31}, given by

$$S_0 = \begin{pmatrix} Id & 0 & 0 \\ 0 & -Id & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_a = \begin{pmatrix} 0 & A_a & B_a \\ A_a^{tr} & 0 & C_a \\ B_a^{tr} & C_a^{tr} & 0 \end{pmatrix}, \quad 1 \leq a \leq 7.$$

The possible choices of A_a, B_a, C_a are further restricted because they must verify that the eigenvalues of S_n are 0 and ± 1 in all normal directions n so that $(S_n)^3 = S_n$. Algebraically, this says

$$\left(\sum_{a=0}^{7} c_a S_a\right)^3 = \left(\sum_{a=0}^{7} c_a^2\right)\left(\sum_{a=0}^{7} c_a S_a\right), \quad \forall c_0 \cdots, c_7 \in \mathbb{R},$$

that an isoparametric hypersurface with four principal curvatures and multiplicity pair $(7, 8)$ enjoys, which simplifies to those equations in (2.3) below, plus a few more not listed (see [**28**, II, p. 45]). This accounts for the possible second fundamental form of the focal manifold and constitutes the first three of the ten defining identities of an isoparametric hypersurface [**28**, I, p. 523]. One must then pin down the third fundamental form of the focal manifold that is convoluted with the second fundamental form in the seven remaining identities.

For instance, one can take $B_a = C_a = 0$ in all F_a and G_a, which is equivalent to Condition A of Ozeki and Takeuchi [**28**, I] to the effect that there is a point $p \in M_+$ at which the shape operators in all normal directions share the same kernel. Then A_a arise from the left or right multiplication of the octonion algebra. Since the two octonion multiplications are inequivalent, it results in two distinct second fundamental forms and three distinct third fundamental forms that give rise to the three inhomogeneous examples in the case when the multiplicity pair is $(7, 8)$. This is the approach taken in [**4**] to give a different proof of a result in [**13**] that states that the existence of a point of Condition A implies that the isoparametric hypersurface is one of the three inhomogeneous ones.

In general, however, there is no known classification of the above quadratic composition formula.

Algebraic geometry comes to the rescue. In this paper, we shall refer to our fairly detailed survey articles [**6**], [**7**] and the references therein for all the background material that we employed in [**1**], [**3**], [**5**] without dwelling much on it, unless necessarily, except to remark that the unified theme in the classification is the notion of normal varieties and Serre's criterion for verifying the normality of a variety, in terms of a subtle codimension 2 test on the generating functions of the ideal of the variety. Its technical side we developed in [**1**], [**3**], [**5**] enabled us to harness the components $p_0, \cdots, p_{m_+}$ of the second fundamental form of the focal manifold M_+ of the smaller codimension $1+m_+$ in the sphere, to gain a good global control over the codimension 2 estimate on the variety carved out by $p_0, \cdots, p_{m_+}$. In fact, an essential step is to study the *singular locus* $\mathscr{S}$ of the (complex) linear system of cones $\mathcal{C}_\lambda$

$$c_0 p_0 + \cdots + c_{m_+} p_{m_+} = 0$$

as $\lambda := [c_0 : \cdots : c_{m_+}]$ sweeps out $\mathbb{C}P^{m_+}$. The codimension 2 estimate gets sharper when we understand better how $p_0, \cdots, p_{m_+}$ cut the singular locus $\mathscr{S}_\lambda$ of the cone $\mathcal{C}_\lambda$, remarking that $\mathscr{S} = \cup_\lambda \mathscr{S}_\lambda$.

In [**1**], [**3**], [**5**], we were able to classify all isoparametric hypersurfaces with four principal curvatures, except for the case when the principal multiplicity pair is $(m_+, m_-) = (7, 8)$, essentially by exploring the cut between $p_0 = p_2 = 0$ and $\mathscr{S}_\lambda$, remarking that, by symmetry, $p_0 = 0$ and $p_1 = 0$ produce the same cut into $\mathscr{S}_\lambda$. Intersection of more varieties needs to be considered for a global codimension 2 estimate in the case when the multiplicity pair is $(7, 8)$, which, however, gets untamed without an effective cutting strategy.

To overcome this obstacle, we introduce in this paper (see Section 3) a notion called r-**nullity**, which generalizes Condition A that is 0-null of Ozeki and Takeuchi, remarking that Condition A is important in the classification of the anomalous cases when the multiplicity pair is $(m_+, m_-) = (3, 4), (4, 5)$, or $(6, 9)$.

In fact, for Serre's codimension 2 test it suffices to consider only those $\mathscr{S}_\lambda$ for which $\lambda = [c_0 : \cdots : c_{m_+}]$ live in the complex hyperquadric

$$c_0^2 + \cdots + c_{m_+}^2 = 0,$$

so that each λ is a 2-plane spanned by an (oriented) orthonormal pair (n_0, n_1) of a normal basis $n_0, n_1, \cdots, n_{m_+}$ with the corresponding components $p_0, p_1, \cdots, p_{m_+}$ [**5**]. Let r be the number

$$r := m_+ - \dim(\mathrm{kernel}(S_{n_0}) \cap \mathrm{kernel}(S_{n_1})).$$

We say a normal basis element $n_l, l \geq 2$, is r-**null** if p_l is identically zero when it is restricted to $\mathscr{S}_\lambda$. We say the normal basis $n_0, n_1, \cdots, n_{m_+}$ is r-null if n_l is r-null for **all** $l \geq 2$.

As we shall see, a normal basis being r-null is the worst case scenario one can encounter in the codimension 2 estimate, since the intersection between each $p_l = 0, l \geq 2$, and $\mathscr{S}_\lambda$ is trivial, and hence contributes nothing to the codimension 2 estimate.

At a first glance, this algebro-geometric definition of r-nullity seems to lack of differential-geometric content. However, we show in Section 3 (see Lemma 3.1) that r-nullity is equivalent to that all the upper left $(m_- - r)$-by-$(m_+ - r)$ blocks of B_a and C_a vanish for $1 \leq a \leq m_+$, so that in particular r-nullity holds if the generic rank of linear combinations of $B_1, \cdots, B_{m_+}$ is r. It is clear now that Condition A is equivalent to that the normal basis is 0-null.

We may assume the isoparametric hypersurface M with multiplicity pair $(m_+, m_-) = (7, 8)$ is not the one constructed by Ozeki and Takeuchi [**28**, I]. Then we can conclude in Sections 5 and 6 (Lemma 5.3 and Proposition 6.1), after a long technical preparation of placing constraints on 1-, 2-, and 3-nullity in Section 4 (with the help of certain codimension 2 estimates given in Appendix I) that the focal manifold M_+ is generically 4-null when we are away from points of Condition A. This enables us to prove in Section 6 the following

Reduction Lemma. *Let M be an isoparametric hypersurface with multiplicities $(m_+, m_-) = (7, 8)$ not constructed by Ozeki and Takeuchi. Given any point $p \in M$ with its unit normal n and any vector v at p tangent to a curvature surface (which is a sphere) of dimension 7, there is a 16-dimensional Euclidean space passing through p, n and v such that it cuts M in a* **homogeneous** *isoparametric hypersurface with multiplicity pair $(m_+, m_-) = (3, 4)$.*

The key ingredient in establishing the reduction lemma is to look back and forth at the "mirror" points [**4**] of a point (x, n) on the unit normal bundle of M_+ and M_-, where M_- is the other focal manifold with larger codimension $1 + m_-$ in the sphere. Here, by the mirror point $(x^\#, n^\#)$ of (x, n) on the unit normal bundle of M_+, and the mirror point (x^*, n^*) of (x, n) on the unit normal bundle of M_-, we mean they are the points

$$(x^\#, n^\#) := (n, x), \quad (x^*, n^*) := ((x+n)/\sqrt{2}, (x-n)/\sqrt{2}).$$

Suffices it to say that the shape operators $S_n, S_{n^\#}$, and S_{n^*} are interlocked (see (6.1), (6.2), (6.4)), so that generic 4-nullity at both x and $x^\#$ enables us to read off many zero blocks of $S_n, S_{n^\#}$, and S_{n^*}, which, when viewed at x^*, fits exactly in the quaternionic framework in [**4**]. Indeed, we have (see (6.5), all counterpart quantities at x^* will be denoted with an extra superscript *)

$$A^*_\alpha = \begin{pmatrix} 0 & 0 \\ 0 & \cdot \end{pmatrix}, \quad B^*_\alpha = \begin{pmatrix} \cdot & 0 \\ 0 & \cdot \end{pmatrix}, \quad C^*_\alpha = \begin{pmatrix} \cdot & 0 \\ 0 & \cdot \end{pmatrix}, \quad 1 \leq \alpha \leq 4;$$
$$A^*_\alpha = \begin{pmatrix} 0 & \cdot \\ \cdot & \cdot \end{pmatrix}, \quad B^*_\alpha = \begin{pmatrix} 0 & \cdot \\ \cdot & \cdot \end{pmatrix}, \quad C^*_\alpha = \begin{pmatrix} 0 & \cdot \\ \cdot & \cdot \end{pmatrix}, \quad 5 \leq \alpha \leq 8,$$

where the lower right blocks are all of size 4-by-4, from which the above reduction lemma follows by investigating how the upper left blocks interact with the remaining blocks through the third fundamental form of M_-.

We are half way home. To determine the remaining blocks of S_n^*, it is more convenient to convert the data to M_+, where now (see (7.1))

$$A_a = \begin{pmatrix} z_a & 0 \\ 0 & w_a \end{pmatrix}, \quad B_a = \begin{pmatrix} 0 & 0 \\ 0 & c_a \end{pmatrix}, \quad C_a = \begin{pmatrix} 0 & 0 \\ 0 & f_a \end{pmatrix}, \quad 1 \le a \le 3,$$
$$A_a = \begin{pmatrix} 0 & \beta_a \\ \gamma_a & \delta_a \end{pmatrix}, \quad B_a = \begin{pmatrix} 0 & d_a \\ b_a & c_a \end{pmatrix}, \quad C_a = \begin{pmatrix} 0 & g_a \\ b_a & f_a \end{pmatrix}, \quad 4 \le a \le 7.$$

An important observation to make is that $(\sqrt{2}c_a, w_a), 1 \le a \le 3$, generate a quadratic composition formula of type $[3,4,8]$. In [**8**], the moduli space of orthogonal multiplications of type $[3,4,p], p \le 12$, is studied; when it is incorporated with the data conversion between x and $x^\#$, we are finally able to specify decisive characteristics of the b_a, c_a, f_a, d_a, g_a blocks, to be presented in Section 7. The driving force for all this to happen is the crucial step that shows the b_a matrices, $4 \le a \le 7$, are generically of rank ≤ 2, so that when we consider the linear combination

$$b(x) := x_1 b_4 + \cdots + x_4 b_7$$

over the polynomial ring $\mathbb{R}[x_1, \cdots, x_4]$, it perfectly fits in the Koszul complex [**15**, p. 423] to let us arrive at the important conclusion that all $b_a, 1 \le a \le 7$, have a common zero column (see Lemma 7.1, Corollary 7.2, and Corollary 7.3). We phrase it in the following context.

Two Universal Properties. *If the isoparametric hypersurface with multiplicity pair* $(m_+, m_-) = (7,8)$ *is not the one constructed by Ozeki and Takeuchi, then at each point of* M_+ *the intersection of kernels of shape operators in all normal directions, or equivalently, of kernels of all* $B_a, 1 \le a \le 7$, *is at least* 1*-dimensional, and moreover, it is* 1*-dimensional at a generic point. Furthermore, the intersection of kernels of all* $B_a^{tr}, 1 \le a \le 7$, *is generically* 2*-dimensional. The statement also holds for* $C_a, 1 \le a \le 7$.

These two properties, pivotal for the classification in this paper, can be seen to hold true for the two isoparametric hypersurfaces constructed by Ferus, Karcher and Münzner through straightforward calculations in Section 2.2 to be given as motivation for subsequent development.

Without plunging into technical details, we point out that, with the characteristic features of $A_a, B_a, C_a, 1 \le a \le 7$, pinpointed, we shall be able to demonstrate in Section 7 that we can come up with a **Clifford frame** over M_- (see (7.20)) in which the second universal property above plays a vital role. In essence, a Clifford frame [**1**], [**2**] gives rise to an 8-dimensional sphere worth of intrinsic isometries of M_- which can be extended to ambient $Spin(9)$ isometries, and hence the hypersurface is one of the two constructed by Ferus, Karcher, and Münzner, if it is not the one constructed by Ozeki and Takeuchi.

It is noteworthy that in recent years there has been much effort to investigate isoparametric foliations on Riemannian manifolds other than the standard spheres, such as exotic spheres [**19**], [**30**], compact manifolds of positive scalar curvature [**31**], complex and quaternionic projective spaces [**10**], [**11**], Damek-Ricci spaces [**9**], and more generally singular foliations on Riemannian manifolds [**17**], [**18**] (and the references therein). Moreover, since isoparametric hypersurfaces form an ideal testing ground to furnish examples and counterexamples, the Yau conjecture on the first eigenvalues of minimal submanifolds in spheres has been mostly established on such hypersurfaces and their focal manifolds [**32**], [**33**], metrics of positive constant scalar curvature have been constructed on products of Riemannian manifolds [**20**], many more stable and unstable examples of Lagrangian submanifolds as Gauss images of such (homogeneous) hypersurfaces in the complex hyperquadrics have been given [**22**], [**23**], and, recently, Hamiltonian non-displaceability of the Gauss images of isoparametric hypersurfaces has been studied [**21**]. (The references are by no means exhaustive.) It is hoped that the completed classification of isoparametric hypersurfaces would spur even more advances far beyond the standard sphere.

2. The basics

2.1. Second fundamental form of a focal manifold. Let M be an isoparametric hypersurface with four principal curvatures in the sphere. Let F be its Cartan-Münzner polynomial of degree g that satisfies [**27**, I]

$$|\nabla F|^2(x) = g^2|x|^{2g-2}, \quad (\Delta F)(x) = (m_- - m_+)g^2|x|^{g-2}/2, \tag{2.1}$$

and let f be the restriction of F to the sphere.

To fix notation, we make the convention that its two focal manifolds are $M_+ := f^{-1}(1)$ and $M_- := f^{-1}(-1)$ with respective codimensions $m_+ + 1 \leq m_- + 1$ in the ambient sphere $S^{2(m_+ + m_-)+1}$ by changing F to $-F$ if necessary. The principal curvatures of the shape operator S_n of M_+ (respectively, M_-) with respect to any unit normal n are $0, 1$ and -1, whose multiplicities are, respectively, m_+, m_- and m_- (respectively, m_-, m_+ and m_+).

On the unit normal sphere bundle UN_+ of M_+, let $(x, n_0) \in UN_+$ be points in a small open set; here $x \in M_+$ and n_0 is normal to the tangents of M_+ at x. We define a smooth orthonormal frame $n_a, e_p, e_\alpha, e_\mu$, where $1 \leq a, p \leq m_+$ and $1 \leq \alpha, \mu \leq m_-$, in such a way that n_a are tangent to the unit normal sphere at n_0, and e_p, e_α and e_μ, respectively, are basis vectors of the eigenspaces E_0, E_+ and E_- of the shape operator S_{n_0}.

The symmetric matrices $S_a := S_{n_a}$ relative to E_+, E_- and E_0 are

$$S_0 = \begin{pmatrix} Id & 0 & 0 \\ 0 & -Id & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_a = \begin{pmatrix} 0 & A_a & B_a \\ A_a^{tr} & 0 & C_a \\ B_a^{tr} & C_a^{tr} & 0 \end{pmatrix} \tag{2.2}$$

for $1 \leq a \leq m_+$, where $A_a : E_- \to E_+$, $B_a : E_0 \to E_+$ and $C_a : E_0 \to E_-$.

Given the second fundamental form $S(X,Y)$, the third fundamental form of M_+ is the symmetric tensor

$$q(X,Y,Z) := (\nabla^\perp_X S)(Y,Z)/3,$$

where $\nabla^\perp$ is the normal connection. Write

$$p_a(X,Y) := \langle S(X,Y), n_a\rangle, \quad q^a(X,Y,Z) = \langle q(X,Y,Z), n_a\rangle$$

for $0 \leq a \leq m_+$. The Cartan-Münzner polynomial F is related to p_a and q^a by the expansion formula of Ozeki and Takeuchi [**28**, I, p. 523]

$$\begin{aligned} F(tx+y+w) &= t^4 + (2|y|^2 - 6|w|^2)t^2 + 8(\sum_{i=0}^{m_+} p_i w_i)t \\ &+ |y|^4 - 6|y|^2|w|^2 + |w|^4 - 2\sum_{i=0}^{m_+} p_i^2 - 8\sum_{i=0}^{m_+} q^i w_i \\ &+ 2\sum_{i,j=0}^{m_+} \langle \nabla p_i, \nabla p_j\rangle w_i w_j, \end{aligned}$$

where $w := \sum_{i=0}^{m_+} w_i n_i$, y is tangential to M_+ at x, $p_i := p_i(y,y)$ and $q^i := q^i(y,y,y)$. Note that our definition of q^i differs from that of Ozeki and Takeuchi by a sign. It follows that the second and third fundamental forms at a single point of M_+ (or M_-) determine the isoparametric family, where the two forms are related by ten rather convoluted equations of Ozeki and Takeuchi [**28**, I, p. 530], of which the first three is a rephrase of the fact that the shape operator S_n in any normal direction n satisfies $(S_n)^3 = S_n$, which implies the following identities, among others [**28**, II, p.45]:

$$A_i A_j^{tr} + A_j A_i^{tr} + 2(B_i B_j^{tr} + B_j B_i^{tr}) = 2\delta_{ij} Id; \tag{2.3a}$$

$$A_i^{tr} A_j + A_j^{tr} A_i + 2(C_i C_j^{tr} + C_j C_i^{tr}) = 2\delta_{ij} Id; \tag{2.3b}$$

$$A_i C_j B_j^{tr} + B_i C_j^{tr} A_j^{tr} + A_j C_i B_j^{tr} \quad \text{is skew-symmetric;} \tag{2.3c}$$

$$C_j B_j^{tr} A_i + A_j^{tr} B_i C_j^{tr} + C_i B_j^{tr} A_j \quad \text{is skew-symmetric;} \tag{2.3d}$$

$$B_j^{tr} A_i C_j + C_j^{tr} A_j^{tr} B_i + B_j^{tr} A_j C_i \quad \text{is skew-symmetric;} \tag{2.3e}$$

$$B_j^{tr}B_i + B_i^{tr}B_j = C_j^{tr}C_i + C_i^{tr}C_j; \tag{2.3f}$$

$$\begin{aligned}&(A_iA_i^{tr} + B_iB_i^{tr})B_j + B_j(B_i^{tr}B_i + C_i^{tr}C_i) + B_iB_j^{tr}B_i +\\ &A_jA_i^{tr}B_i + A_iA_j^{tr}B_i + B_iC_i^{tr}C_j + B_iC_j^{tr}C_i = B_j;\end{aligned} \tag{2.3g}$$

$$C_i^{tr}A_i^{tr}B_i + B_i^{tr}A_iC_i = 0. \tag{2.3h}$$

Lemma 49 [**1**, p. 64] ensures that we can assume

$$B_1 = C_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}, \tag{2.4}$$

where σ is a nonsingular diagonal matrix of size r-by-r with r the rank of B_1, and A_1 is of the form

$$A_1 = \begin{pmatrix} I & 0 \\ 0 & \Delta \end{pmatrix}, \tag{2.5}$$

where $\Delta = \mathrm{diag}(\Delta_1, \Delta_2, \Delta_3, \cdots)$ is of size r-by-r, in which $\Delta_1 = 0$ and $\Delta_i, i \geq 2$, are nonzero skew-symmetric matrices expressed in the block form $\Delta_i = \mathrm{diag}(\Theta_i, \Theta_i, \Theta_i, \cdots)$ with Θ_i a 2-by-2 matrix of the form

$$\begin{pmatrix} 0 & f_i \\ -f_i & 0 \end{pmatrix}$$

for some $0 < f_i < 1$, where the block of σ corresponding to $\Delta_1 = 0$ is $I/\sqrt{2}$.

Definition 2.1. *We call a normal basis $n_0, n_1, n_2, \cdots, n_{m_+}$ (or simply the pair (n_0, n_1))* **normalized** with **spectral data** (σ, Δ) if S_0 and $S_a, 1 \leq a \leq m_+$, are given in (2.2) satisfying (2.4) and (2.5).

Remark 2.1. *The geometric meaning of the rank r of B_1 is that $m_+ - r$ is the dimension of the intersection of the kernels of the two shape operators S_0 and S_1.*

Corollary 2.1. *Let $(m_+, m_-) = (7, 8)$. Let an integer $0 \leq r \leq 7$ be the rank of B_1 of size 8-by-7, which is normalized as in (2.4).*

- **(1):** *Assume $r > 0$. The first $8-r$ rows of B_a and C_a are zero for at most one index a between 2 and 7 when $r = 2$, and at most three indexes a when $r = 4$. No other r are possible.*
- **(2):** *Assume $r = 0$. Away from points of Condition A on M_+, if B_2 (vs. C_2) is of rank 2, then no index a between 3 and 7 can make the first six rows of B_a (vs. C_a) zero . Moreover, if B_2 (vs. C_2) is of rank 4, then at most two indexes a between 3 and 7 can make the first four rows of B_a (vs. C_a) zero.*

Proof. To prove the first statement, let A_1 and $B_1 = C_1$ be normalized. Assume without loss of generality that the first $8 - r$ rows of B_a and C_a are zero. Write

$$A_a = \begin{pmatrix} \alpha_a & \beta_a \\ \gamma_a & \delta_a \end{pmatrix}, \quad B_a = \begin{pmatrix} 0 & 0 \\ b_a & c_a \end{pmatrix}, \quad C_a = \begin{pmatrix} 0 & 0 \\ e_a & f_a \end{pmatrix} \tag{2.6}$$

for $2 \leq a \leq 7$, where δ_a, c_a, f_a are of size r-by-r.

(2.3a) applied to A_i and $j = 1$ gives

$$\alpha_i = -\alpha_i^{tr}, \quad \gamma_i^{tr} = \beta_i \Delta,$$

while (2.3c) gives

$$\beta_i \sigma^2 = 0,$$

so that $\beta_i = \gamma_i = 0$.

Suppose there are k indexes $i_1, \cdots, i_k$ between 2 and 7 satisfying (2.6). Then it follows from (2.3a) applied to $A_i, A_j, 2 \leq i, j$, that

$$\alpha_{i_s}\alpha_{i_t} + \alpha_{i_t}\alpha_{i_s} = -2\delta_{st} I. \tag{2.7}$$

Meanwhile, $\alpha_{i_1}, \cdots, \alpha_{i_k}$ are linearly independent; or else a suitable linear combination of them will make, say, $\alpha_{i_1} = 0$ after a basis change, which contradicts (2.7). Therefore, the k $(8 - r)$-by-$(8 - r)$ matrices $\alpha_{i_1}, \cdots, \alpha_{i_k}$ make $\mathbb{R}^{8-r}$ into a Clifford C_k-module, so that $\dim(C_k)$ divides $8 - r$. We conclude by the classification table of C_k that $k = 1$, i.e., there is only one index a between 2 and 7 when $r = 2$ because only $\dim(C_1) = 2$ divides $6 = 8 - r$. Likewise, $k \leq 3$ when $r = 4$, i.e., there are at most three indexes a between 2 and 7 when $r = 4$ because $\dim(C_3) = 4$ divides $4 = 8 - r$ while $\dim(C_4) = 8$. This proves item (1).

When $r = 0$, one of the pairs $(B_2, C_2), \cdots, (B_7, C_7)$ is nonzero, say $(B_2, C_2) \neq 0$, for lack of Condition A. We may swap n_1 and n_2 so that the old n_2 is now the new $n_1^{'}$ with the new $r' \neq 0$, while the old n_1 is now the new $n_2^{'}$ with the new $B_{2'} = C_{2'} = 0$. We apply item (1) to this new indexing to conclude that there is at most one index a' between $2'$ and $7'$ for which the first six rows of $B_{a'}$ and $C_{a'}$ are zero when $r' = 2$, namely, $a' = 2$ itself. That is, in terms of the old indexing, no a between 3 and 7 can make the first 6 rows of B_a and C_a zero when the old B_2 is of rank 2.

Meanwhile, the same argument applies to the new indexing to give at most three indexes $a' \geq 2$ to make the first four rows of $B_{a'}$ and $C_{a'}$ zero when $r' = 4$, namely, $a' = 2$ and two other indexes. That is, in terms of the old indexing, at most two indexes a between 3 and 7 can make the first four rows of B_a and C_a zero when the old B_2 is of rank 4. This proves the second statement.

q.e.d.

2.2. A motivational calculation. Let $\rho_1, \cdots, \rho_7$ be a representation of the (anti-symmetric) Clifford algebra C_7 on $\mathbb{R}^{16}$. Set

$$\begin{aligned} P_0 &: (c,d) \mapsto (c,-d),\\ P_1 &: (c,d) \mapsto (d,c),\\ P_{1+i} &: (c,d) \mapsto (\rho_i(d), -\rho_i(c)), \quad 1 \le i \le 7, \end{aligned}$$

over $\mathbb{R}^{32} = \mathbb{R}^{16} \oplus \mathbb{R}^{16}$. $P_0, P_1, \cdots, P_8$ form a representation of the (symmetric) Clifford algebra C_9' on $\mathbb{R}^{32}$.

Following our convention, we denote by M_- the focal manifold in each of the two examples constructed by Ferus, Karcher, and Mïzner on which the Clifford action acts. It is well known [**16**] that M_- can be realized as the Clifford-Stiefel manifold. Namely,

$$\begin{aligned} M_- = \{(\zeta,\eta) \in S^{31} \subset \mathbb{R}^{16} \times \mathbb{R}^{16} :\\ |\zeta| = |\eta| = 1/\sqrt{2}, \zeta \perp \eta, \rho_i(\zeta) \perp \eta, i = 1, \cdots, 7\}. \end{aligned}$$

At $(\zeta,\eta) \in M_-$, the normal space is

$$N^* = \operatorname{span}(\epsilon_0 := P_0((\zeta,\eta)), \cdots, \epsilon_8 := P_8((\zeta,\eta))).$$

E_0^*, the 0-eigenspace of the shape operator $S_0^* := S_{\epsilon_0}^*$, is

$$E_0^* = \operatorname{span}(\epsilon_9 := P_1P_0((\zeta,\eta)), \cdots, \epsilon_{16} := P_8P_0((\zeta,\eta))).$$

$E_\pm^*$, the ± 1-eigenspaces of S_0^*, are

$$E_\pm^* := \{X : P_0(X) = \mp X, X \perp N^*\}.$$

Since E_+^* (respectively, E_-^*) consists of vectors of the form $(0,d) \in \mathbb{R}^{32}$ (respectively, $(f,0) \in \mathbb{R}^{32}$), we obtain

$$\begin{aligned} E_+^* &= \{(0,d) : d \perp \zeta, d \perp \eta, d \perp \rho_i(\zeta), \forall i\},\\ E_-^* &= \{(f,0) : f \perp \zeta, f \perp \eta, f \perp \rho_i(\eta), \forall i\}. \end{aligned}$$

The shape operator S_α^* at $(\zeta,\eta) \in M_-$ in the normal direction $\epsilon_a \in N^*$ is

$$S_\alpha^*(X,Y) = -\langle P_\alpha(X), Y\rangle, \quad 0 \le \alpha \le 8.$$

For illustrating purpose, let us look at the representation

$$\rho_i : \mathbb{O} \oplus \mathbb{O} \to \mathbb{O} \oplus \mathbb{O}, \quad \rho_i : (x,y) \mapsto (xe_i, ye_i), \quad 1 \le i \le 7,$$

where

$$(e_0, e_1, \cdots, e_7) := (1, i, j, k, \epsilon, \epsilon i, \epsilon j, \epsilon k)$$

are the standard basis elements of the octonion algebra $\mathbb{O}$.

Let us choose

$$\zeta = (e_0, e_1)/2, \quad \eta = (e_3, e_4)/2.$$

We calculate to see

$$\begin{aligned} E_+^* &= \{((0,0),(u,v)) : u = e_1 v, v \perp e_2\},\\ E_-^* &= \{((x,y),(0,0)) : x = e_3(e_2 y), y \perp e_1\}. \end{aligned}$$

Therefore, the 7-by-7 A_α^*-block of S_α^* reads

$$A_\alpha^* = \left(S_\alpha^*(X_a, Y_p)\right) = \left(-\langle P_\alpha(X_a), Y_p\rangle\right), \quad 0 \leq \alpha \leq 8,$$

where X_a, Y_p are orthonormal basis elements in E_+^* and E_-^*, respectively, which can be chosen to be, respectively,

$$((0,0),(e_1e_b,e_b))/\sqrt{2}, \;\; b \neq 2, \quad ((e_3(e_2e_q),e_q),(0,0))/\sqrt{2}, \;\; q \neq 1,$$

arranged, in order, to be $X_a, 1 \leq a \leq 7$, and $Y_p, 1 \leq p \leq 7$.

As said in the introduction, this calculation is conducted at $(x^*, n_0^*) := ((\zeta,\eta),\epsilon_0)$ on the unit normal bundle of M_-, and we can convert it to its mirror point (x, n_0) on the unit normal bundle of M_+, so that in fact the data A_α^* are converted to the seven 8-by-7 matrices

$$B_a := \left(S_\alpha^*(X_a, Y_p)\right), \quad 1 \leq a \leq 7,$$

where B_a is the B-block of the shape operator S_a, given in (2.2), at $x \in M_+$ in the normal direction $n_a := X_a, 1 \leq a \leq 7$. (See (6.1), (6.4) for the conversion formulae.) The upshot is the following:

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & J & 0 \\ 0 & 0 & 0 & -J \end{pmatrix}, B_3 = 0, B_4 = \begin{pmatrix} 0 & 0 & L & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix}$$

$$B_5 = \begin{pmatrix} 0 & 0 & K & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -J & 0 & 0 \end{pmatrix}, B_6 = \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \\ 0 & -L & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B_7 = \begin{pmatrix} 0 & 0 & 0 & J \\ 0 & 0 & 0 & 0 \\ 0 & -K & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad L := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Here, each row is of size 2, and the first column is of size 1 and the remaining columns are of size 2.

Note that x is not of Condition A, and all B_a have a common zero column and all B_a^{tr} have two common zero columns. This is the content mentioned in the two universal properties in the introduction. We shall see in the next section that the basis associated with $B_1, \cdots, B_7$ is 4-null, a notion briefly introduced in the introduction. This example shall be our prototype to keep in mind.

3. r-nullity

3.1. The layout. To fix notation and for the reader's convenience, let us first summarize the layout in [**3**], [**5**] of the crucial codimension 2 estimate in the case when the principal multiplicity pair of the isoparametric hypersurface is not $(7,8)$. We then point out the insufficiency of this approach and the need for a notion more general than Condition

A of Ozeki and Takeuchi, when the principal multiplicity pair of the isoparametric hypersurface is $(m_+, m_-) = (7, 8)$.

Recall that on M_+ we denote by $S_0, \cdots, S_{m_+}$ the shape operators in the normal directions $n_0, \cdots, n_{m_+}$, and by $p_0, \cdots, p_{m_+}$ the corresponding components of the second fundamental form.

We agree that $\mathbb{C}^{2m_-+m_+}$ consists of points (u, v, w) with coordinates u_α, v_μ and w_p, where $1 \leq \alpha, \mu \leq m_-$ and $1 \leq p \leq m_+$. For $0 \leq k \leq m_+$, let

$$V_k := \{(u, v, w) \in \mathbb{C}^{2m_-+m_+} : p_0(u, v, w) = \cdots = p_k(u, v, w) = 0\}$$

be the variety carved out by $p_0, \cdots, p_k$. We want to estimate the dimension of the subvariety $\mathscr{J}_k$ of $\mathbb{C}^{2m_-+m_+}$, where

$$\mathscr{J}_k := \{(u, v, w) \in \mathbb{C}^{2m_-+m_+} : \text{rank of Jacobian of } p_0, \cdots, p_k < k + 1\}.$$

$p_0, \cdots, p_k$ give rise to a linear system of cones $\mathcal{C}_\lambda$ in $\mathbb{C}^{2m_-+m_+}$ defined by

$$c_0 p_0 + \cdots + c_k p_k = 0$$

with

$$\lambda := [c_0 : \cdots : c_k] \in \mathbb{C}P^k.$$

The singular subvariety of $\mathcal{C}_\lambda$ is

$$(3.1) \quad \mathscr{S}_\lambda := \{(u, v, w) \in \mathbb{C}^{2m_-+m_+} : (c_0 S_0 + \cdots + c_k S_k) \cdot (u, v, w)^{tr} = 0\}.$$

We have

$$\mathscr{J}_k = \bigcup_{\lambda \in \mathbb{C}P^k} \mathscr{S}_\lambda.$$

Set

$$J_k := V_k \cap \mathscr{J}_k = \bigcup_{\lambda \in \mathbb{C}P^k} (V_k \cap \mathscr{S}_\lambda).$$

J_k is where the Jacobian of $p_0, \cdots, p_k$ fails to be of rank $k + 1$ on the variety V_k.

We wish to establish the codimension 2 estimate

$$(3.2) \qquad \dim(J_k) \leq \dim(V_k) - 2,$$

for all $k \leq m_+ - 1$, to verify that $p_0, p_1, \cdots, p_{m_+}$ form a regular sequence.

We first estimate the dimension of $\mathscr{S}_\lambda$. We established in [**5**] that it suffices to consider those λ sitting in the hyperquadric

$$(3.3) \qquad \mathcal{Q}_{k-1} := \{[c_0 : \cdots : c_k] \in \mathbb{C}P^k : c_0^2 + \cdots + c_k^2 = 0\}.$$

Recall the following [**5**, Remark 2, p. 484].

Convention 3.1. *For each $\lambda = [c_0 : \cdots : c_k] \in \mathcal{Q}_{k-1}$, we choose $\tilde{n}_0$ and $\tilde{n}_1$ as follows. Decompose $n := c_0 n_0 + \cdots + c_k n_k$ into its real and imaginary parts $n = \alpha + \sqrt{-1}\beta$. Define $\tilde{n}_0$ and $\tilde{n}_1$ by performing the Gram-Schmidt process on α and β. Then normalize the shape operators $S_{\tilde{n}_0}, S_{\tilde{n}_1}$ as in* (2.4) *and* (2.5)*, which results in a 2-frame $(\tilde{n}_0, \tilde{n}_1)$ that* *varies smoothly with λ. Note that λ can be interpreted as the oriented real 2-plane spanned by $n_{\tilde{0}}$ and $n_{\tilde{1}}$.*

We denote the rank of the matrix $B_{\tilde{1}}$ associated with $S_{\tilde{n}_1}$ by r_λ. Recall from Remark 2.1 that $m_+ - r_\lambda$ is the dimension of the intersection of the kernel spaces of $S_{\tilde{n}_0}$ and $S_{\tilde{n}_1}$.

When it is necessary, we will extend $\tilde{n}_0$ and $\tilde{n}_1$ to an orthonormal basis $\tilde{n}_0, \tilde{n}_1, \cdots, \tilde{n}_{m_+}$ with the corresponding shape operators $S_{\tilde{0}} := S_{\tilde{n}_0}, S_{\tilde{1}} := S_{\tilde{n}_1}, \cdots, S_{\tilde{m}_+} := S_{\tilde{n}_{m_+}}$ and components $p_{\tilde{0}}, p_{\tilde{1}}, \cdots, p_{\tilde{m}_+}$ of the second fundamental form.

The convention facilitates the dimension estimate for $\mathscr{S}_\lambda$. Indeed, the defining equation of $\mathscr{S}_\lambda$ can now be written as

$$(S_{\tilde{1}} - \iota_\lambda S_{\tilde{0}}) \cdot (x, y, z)^{tr} = 0 \tag{3.4}$$

after a basis change for some complex number ι_λ. We decompose x, y, z into $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2)$ with $x_2, y_2, z_2 \in \mathbb{C}^{r_\lambda}$. We have

$$\begin{gathered} x_1 = -\iota_\lambda y_1, \qquad y_1 = \iota_\lambda x_1, \\ -\Delta x_2 + \sigma z_2 = -\iota_\lambda y_2, \qquad \Delta y_2 + \sigma z_2 = \iota_\lambda x_2, \\ \Delta(x_2 + y_2) = 0. \end{gathered} \tag{3.5}$$

It follows from the first pair of equations in (3.5) that either $x_1 = y_1 = 0$, or both are nonzero with $\iota_\lambda = \pm\sqrt{-1}$. In both cases, by the second pair of equations in (3.5), we have

$$(\Delta^2 - \iota_\lambda^2 I)x_2 = (\Delta - \iota_\lambda I)\sigma z_2, \qquad (\Delta^2 - \iota_\lambda^2 I)y_2 = -(\Delta - \iota_\lambda I)\sigma z_2, \tag{3.6}$$

which together with the third equation in (3.5) imply that $x_2 = -y_2$, and so z_2 can be solved in terms of x_2 by the second pair of equations in (3.5). (Note that conversely $x_2 = -y_2$ can be solved in terms of z_2 when $\iota_\lambda \neq \pm f_i\sqrt{-1}$ for all i and any real $0 < f_i < 1$, so that z can be chosen to be a free variable.) Thus either $x_1 = y_1 = 0$, in which case

$$\dim(\mathscr{S}_\lambda) = m_+,$$

or both x_1 and y_1 are nonzero, in which case $y_1 = \pm\sqrt{-1}x_1$, where x_1 is a free variable, x_2 and y_2 depend linearly on z_2 and z is a free variable. Hence,

$$\dim(\mathscr{S}_\lambda) = m_+ + m_- - r_\lambda. \tag{3.7}$$

Since eventually we must estimate the dimension of

$$\bigcup_{\lambda \in \mathcal{Q}_{k-1}} (V_k \cap \mathscr{S}_\lambda),$$

the essential part of J_k for the codimension 2 test, we introduced the first cut of V_k into $\mathscr{S}_\lambda$ by

$$0 = p_{\tilde{0}} = \sum_\alpha (x_\alpha)^2 - \sum_\mu (y_\mu)^2. \tag{3.8}$$

We substitute $y_1 = \pm\sqrt{-1}x_1$ and x_2 and y_2 in terms of z_2 into $p_{\tilde{0}} = 0$ to deduce

$$0 = (x_1)^2 + \cdots + (x_{m_- - r_\lambda})^2;$$

hence $p_{\tilde{0}} = 0$ cuts $\mathscr{S}_\lambda$ to reduce the dimension by 1, i.e., by (3.7),

$$\dim(V_k \cap \mathscr{S}_\lambda) \leq m_+ + m_- - r_\lambda - 1. \tag{3.9}$$

Consider the incidence space

$$\mathcal{I}_k := \{(x, \lambda) \in \mathbb{C}^{2m_- + m_+} \times \mathcal{Q}_{k-1} : x \in \mathscr{S}_\lambda \cap V_k\}. \tag{3.10}$$

Let π_1 and π_2 be the restriction to $\mathcal{I}_k$ of the standard projections from $\mathbb{C}^{2m_- + m_+} \times \mathcal{Q}_{k-1}$ onto the first and second factors. We see

$$\pi_1(\mathcal{I}_k) = \bigcup_{\lambda \in \mathcal{Q}_{k-1}} (V_k \cap \mathscr{S}_\lambda).$$

Moreover, if we stratify $\mathcal{Q}_{k-1}$ into locally closed sets (i.e., Zariski open sets in their respective closures)

$$\mathcal{L}_j := \{\lambda \in \mathcal{Q}_{k-1} : r_\lambda = j\}, \tag{3.11}$$

then

$$\mathcal{W}_j := \pi_1 \pi_2^{-1}(\mathcal{L}_j)$$

stratify

$$\bigcup_{\lambda \in \mathcal{Q}_{k-1}} (V_k \cap \mathscr{S}_\lambda).$$

We thus obtain, by (3.9),

$$\begin{aligned} \dim(\mathcal{W}_j) &\leq \dim(\pi_2^{-1}(\mathcal{L}_j)) \leq \max_{\lambda \in \mathcal{L}_j}(\dim(V_k \cap \mathscr{S}_\lambda)) + \dim(\mathcal{L}_j) \\ &\leq (m_+ + m_- - 1 - j) + \dim(\mathcal{L}_j) \end{aligned} \tag{3.12}$$

On the other hand, since V_k is cut out by $k + 1$ equations, we have

$$\dim(V_k) \geq m_+ + 2m_- - k - 1. \tag{3.13}$$

Therefore, the *a priori* codimension 2 estimate holds true over $\mathcal{L}_j$ when

$$m_- \geq 2k + 1 - j - c_j, \tag{3.14}$$

where

$$c_j := \text{the codimension of } \mathcal{L}_j \text{ in } \mathcal{Q}_{k-1}. \tag{3.15}$$

3.2. r-nullity. Note that we only utilized cutting $\mathscr{S}_\lambda$ by $p_{\tilde{0}} = 0$ to derive the coarse upper bound in (3.9) and lower bound in (3.14). The lower bound is too rough to be effective when the multiplicity pair is $(7, 8)$. A better upper or lower bound will be achieved if we can obtain further nontrivial cuts into $\mathscr{S}_\lambda$ by other $p_{\tilde{a}} = 0, a \geq 1$.

As a matter of fact, $p_{\tilde{1}} = 0$ results in the same cut on $\mathscr{S}_\lambda$ as $p_{\tilde{0}} = 0$. This follows by the symmetry of (3.4) so that we can switch the roles of $S_{\tilde{0}}$ and $S_{\tilde{1}}$. Therefore, nontrivial new cuts can only be obtained by $p_{\tilde{a}} = 0$ for $a \geq 2$.

On the other hand, the worst case scenario is that $p_{\tilde{a}}$ annihilate $\mathscr{S}_\lambda$ for all $a \geq 2$, in which case no more cuts other than $p_{\tilde{0}} = 0$ can be introduced and (3.14) is the best possible lower bound. We categorize this worst case in the following definition in the language of (3.5) and (3.6).

Definition 3.1. *Given a normal basis $n_0, \cdots, n_{m_+}$ at a point of M_+ with the usual $A_i, B_i, C_i, 1 \leq i \leq m_+$, and the normalization as in* (2.2), (2.4) *and* (2.5) *with r the rank of both B_1 and C_1, let $p_0, \cdots, p_{m_+}$ be the associated components of the second fundamental form.*

Let $\mathbb{C}^{m_-} \simeq \mathbb{C}E_+$, $\mathbb{C}^{m_-} \simeq \mathbb{C}E_-$ and $\mathbb{C}^{m_+} \simeq \mathbb{C}E_0$ be parametrized by x, y and z respectively, where E_+, E_- and E_0 are the eigenspaces of S_0 with eigenvalues $1, -1$ and 0, respectively. Let $x := (x_1, x_2), y := (y_1, y_2)$ and $z := (z_1, z_2)$ with $x_2, y_2, z_2 \in \mathbb{C}^r$.

We say a normal basis element $n_l, l \geq 2$, is r**-null** *if p_l is identically zero when we restrict it to the linear constraints*

$$y_1 = \iota x_1, \quad y_2 = -x_2, \quad z_2 = \sigma^{-1}(\Delta + \iota I)x_2, \quad \iota = \pm\sqrt{-1}. \tag{3.16}$$

We say the normal basis, always understood to be with the normalization (2.2), (2.4) *and* (2.5), *is r-null if* **all** *its basis elements $n_l, l \geq 2$, are r-null.*

Lemma 3.1. *Conditions as given in the above definition, a normal basis element n_l is r-null if and only if the upper left $(m_- - r)$-by-$(m_+ - r)$ block of B_l and C_l of S_l are zero.*

Proof. Suppose n_l is r-null. Then p_l restricted to the linear constraint in the definition is

$$p_l = \sum_{\alpha=1, p=1}^{m_- - r, m_+ - r} (S^l_{\alpha p} + \iota T^l_{\alpha p}) x_\alpha z_p + \text{ other terms}, \tag{3.17}$$

where

$$S^l_{\alpha p} := \langle S(X_\alpha, Z_p), n_l \rangle, \quad T^l_{\alpha p} := \langle S(Y_\alpha, Z_p), n_l \rangle$$

for some orthonormal basis X_α, Y_α, Z_p of E_+, E_-, E_0, respectively. Therefore,

$$S^l_{\alpha p} = T^l_{\alpha p} = 0 \tag{3.18}$$

for $1 \leq \alpha \leq m_- - r$ and $1 \leq p \leq m_+ - r$.

Conversely, suppose (3.18) is true, from which we first derive some identities. Let A_1, B_1, C_1 be normalized as in (2.4) and (2.5). Write

$$A_l := \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad B_l := \begin{pmatrix} 0 & d \\ b & c \end{pmatrix}, \quad C_l := \begin{pmatrix} 0 & g \\ e & f \end{pmatrix},$$

where δ, c, f are of size r-by-r. (2.3c) applied to $i = l$ and $j = 1$, with the property

$$\sigma\Delta = \Delta\sigma,$$

gives

$$\beta - d\sigma^{-1}\Delta + g\sigma^{-1} = 0, \tag{3.19a}$$

while (2.3d) gives

$$d\sigma^{-1} + \gamma^{tr} + g\sigma^{-1}\Delta = 0. \tag{3.19b}$$

Meanwhile, (2.3f) arrives at

$$b = e, \quad c^{tr}\sigma + \sigma c = f^{tr}\sigma + \sigma f. \tag{3.19c}$$

In particular, writing

$$h := c - f,$$

we obtain

$$\sigma h + h^{tr}\sigma = 0. \tag{3.19d}$$

Now, we can rewrite (3.19d) as

$$\sigma(h\sigma^{-1} + \sigma^{-1}h^{tr})\sigma = 0,$$

from which we see

$$h\sigma^{-1} + \sigma^{-1}h^{tr} = 0. \tag{3.19e}$$

Next, (2.3e) asserts

$$\sigma(\delta + \delta^{tr})\sigma - \sigma\Delta h + h^{tr}\sigma\Delta = 0, \tag{3.19f}$$

or equivalently,

$$\delta + \delta^{tr} - \Delta h\sigma^{-1} + \sigma^{-1}h^{tr}\Delta = 0,$$

or if we employ (3.19e), which is $h\sigma^{-1} = -\sigma^{-1}h^{tr}$, we can rewrite it as

$$\delta + \delta^{tr} - h\sigma^{-1}\Delta + \sigma^{-1}\Delta h^{tr} = 0. \tag{3.19g}$$

In general,

$$p_l/2 = x^{tr}A_l y + x^{tr}B_l z + y^{tr}C_l z;$$

setting $x = x_1 + x_2, y = y_1 + y_2, z = z_1 + z_2$, and employing (3.16), we can rewrite it in terms of the independent variables x_1, x_2 and z_1 as

$$
\begin{aligned}
&p_l/2\\
&= x_1^{tr}(-\beta + \tau\gamma^{tr} + d\sigma^{-1}(\Delta + \tau I) + \tau g\sigma^{-1}(\Delta + \tau I))x_2\\
&+ x_2^{tr}(b - e)z_1 + x_2^{tr}(-\delta - \delta^{tr} + (c - f)\sigma^{-1}(\Delta + \tau I)\\
&+ ((c - f)\sigma^{-1}(\Delta + \tau I))^{tr})x_2/2\\
&= x_1^{tr}((-\beta + d\sigma^{-1}\Delta - g\sigma^{-1}) + \tau(d\sigma^{-1} + \gamma^{tr} + g\sigma^{-1}\Delta))x_2\\
&+ x_2^{tr}(b - e)z_1 + x_2^{tr}((-(\delta + \delta^{tr}) + h\sigma^{-1}\Delta - \sigma^{-1}\Delta h^{tr})\\
&+ \tau(h\sigma^{-1} + \sigma^{-1}h^{tr}))x_2/2\\
&= 0
\end{aligned}
\tag{3.20}
$$

by (3.19a), (3.19b), (3.19c), (3.19e), (3.19g). q.e.d.

Corollary 3.1. *Condition A of Ozeki and Takeuchi is equivalent to that all normal bases are* 0*-null at a point of Condition A.*

Proof. The statement follows immediately from Lemma 3.1. q.e.d.

Remark 3.1. *The calculation in Section* 2.2 *shows that there the normal basis associated with the displayed* $B_1, \cdots, B_7$ *is* 4*-null.*

Corollary 3.2. *Let* $(m_+, m_-) = (7, 8)$*. Let* $\lambda \in \mathcal{Q}_6$ *be given in* (3.3) *with* $S_{\tilde 0}$ *and* $S_{\tilde 1}$ *normalized as in Convention* 3.1 *and* (2.4) *and* (2.5)*. Suppose*

$$r := \sup_{\lambda \in \mathcal{Q}_6} r_\lambda.$$

Then the upper left $(m_- - r)$*-by-*$(m_+ - r)$ *corner of* $B_{\tilde l}$ *and* $C_{\tilde l}$ *of* $S_{\tilde l}$ *are zero,* $2 \le l \le 7$*, for all* $\lambda \in \mathcal{Q}_6$*. That is, the basis elements* $\tilde n_l, l \ge 2$*, are* r*-null.*

Proof. Pick a generic $\lambda_0 \in \mathcal{Q}_6$ at which $r_{\lambda_0} = r$. Without loss of generality, at λ_0, the 2-plane spanned by the frame $(\tilde n_0, \tilde n_1)$, let us consider $\tilde n_2$ with $S_{\tilde 0}$ and $S_{\tilde 1}$ normalized as usual by (2.4) and (2.5). Set

$$B_{\tilde 2} = \begin{pmatrix} a & d \\ b & c \end{pmatrix}, \quad C_{\tilde 2} = \begin{pmatrix} h & g \\ e & f \end{pmatrix},$$

where c and f are of size r-by-r. We show $a = h = 0$.

Let $e_1, \cdots, e_8$ be the standard (column) basis vectors of $\mathbb{R}^8$. Consider the 8-by-7 $B(\theta) := \cos(\theta)B_{\tilde 1} + \sin(\theta)B_{\tilde 2}$. We have

$$B(\theta) = \begin{pmatrix} \sin(\theta)a & \sin(\theta)d \\ \sin(\theta)b & \cos(\theta)\sigma + \sin(\theta)c \end{pmatrix}.$$

For a generic choice of θ, the last r columns of $B(\theta)$ are linearly independent, as is so for those of σ at $\theta = 0$, which span the column space V^θ of $B(\theta)$ of dimension r. Note that, dividing out by $\sin(\theta)$, each of the first $7-r$ column vectors of $B_{\tilde{2}}$ belongs to V^θ. Letting θ approach zero, we see these $7-r$ vectors also belong to V^0, which is spanned by $e_{9-r}, \cdots, e_8$. It follows that $a=0$. Likewise, $h=0$. This shows that the statement is true for all generic $\lambda \in \mathcal{Q}_6$. Hence, it is true for all $\lambda \in \mathcal{Q}_6$ by passing to the limit. q.e.d.

Remark 3.2. *The arguments in Corollary* 3.2 *can be strengthened as follows. Notation as in Corollary* 3.2, *suppose* $\lambda(\theta), 0 \leq \theta \leq 1$, *is an analytic curve in* $\mathcal{Q}_6$ *with* λ *spanned by an oriented frame* $(\tilde{n}_0, \tilde{n}(\theta))$, *where* $\tilde{n}(\theta) \perp \tilde{n}_0$ *with* $\tilde{n}(0) = \tilde{n}_1$. *Denote by* $B(\theta)$ *the* B*-block of the shape operator* $S_{\tilde{n}(\theta)}$ *and suppose* $B(0)$ *is normalized as in* (2.4) *with rank* r.

Assume the rank of $B(\theta) = r$ *for generic* θ. *Then generic* $B(\theta)$ *has the property that the last* r *columns are independent as is the case for* $B(0)$. *Let us denote the matrix of the first* $7-r$ *columns of* $B(\theta)$ *by*

$$\begin{pmatrix} a(\theta) \\ b(\theta) \end{pmatrix},$$

where $a(\theta)$ *is of size* $(8-r)$*-by-*$(7-r)$.

Suppose $a(\theta) \neq 0$. *It is well-known in analytic curve theory that we can choose the Frenet frame* $\tilde{n}_2, \cdots, \tilde{n}_7$ *such that*

$$\tilde{n}(\theta) = c_1(\theta)\tilde{n}_1 + \cdots + c_7(\theta)\tilde{n}_7 \tag{3.21}$$

for some analytic functions $c_1, \cdots, c_7$, *where*

$$c_1(0) = 1, \quad c_l(\theta) = \theta^{k_l} d_l(\theta), \quad d_l(0) \neq 0, \quad k_2 < \cdots < k_7, \quad l \geq 2,$$

where $\tilde{n}_2$ *is tangent to* $\tilde{n}(\theta)$ *with contact order* k_2 *at* $\theta = 0$.

Dividing through by θ^{k_2}, *it follows that each column of*

$$\begin{pmatrix} a(\theta) \\ b(\theta) \end{pmatrix} / \theta^{k_2}$$

lives in the vector space V^θ *that converges to* V^0 *spanned by* $e_{9-r}, \cdots, e_8$, *as* θ *approaches zero. This implies that* $\tilde{n}_2$ *is* r*-null as in the preceding corollary. Note that the rank of the matrix*

$$B(\theta) := c_1(\theta) B_{\tilde{n}_1} + c_2(\theta) B_{\tilde{n}_2} \tag{3.22}$$

is $= r$ *generically.*

For simplicity of exposition, we assume $r = 2$ *now, though it is true for any* r. *Dividing through by* $c_1(\theta)$, *we may assume* $c_1(\theta) = 1$ *in* (3.22), *as far as the rank of* $B(\theta)$ *is concerned. Then the matrix* $B(\theta)$, *of rank*

2, takes the form

$$B(\theta) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & a_1 & b_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & a_6 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & 1+\alpha & \beta \\ d_1 & d_2 & d_3 & d_4 & d_5 & \gamma & 1+\delta \end{pmatrix},$$

where all the variables $a, b, c, d, \alpha, \beta, \gamma, \delta$ are Taylor series with initial terms of the form θ^{k_2} about $\theta = 0$. We leave it as a simple observation to see that if either the lower left or the upper right block of the matrix is of rank 2, then the other is zero; thus $B_{\tilde{n}_2}$, being $B(\theta)$ with the two diagonal 1s removed, is of rank ≤ 2. Otherwise, the upper and lower blocks are both of rank ≤ 1, in which case we may assume $a_i = b_i = c_j = d_j = 0$ for $1 \leq i \leq 5, 1 \leq j \leq 4$, via row and column reductions. Then with $c_5, d_5, a_6, b_6, \alpha$ and β all essentially being constant multiples of θ^{k_2}, it is readily seen that the 3-by-3 lower right diagonal determinant being zero (because $B(\theta)$ is of rank 2) implies that the 3-by-3 determinant without the two diagonal 1s vanishes as well, i.e., that $B_{\tilde{n}_2}$ is again of rank ≤ 2. (For instance, we may divide by θ^{k_2} and let θ go to infinity.) Consequently, the analytic

$$\cos(\theta)B_{\tilde{n}_1} + \sin(\theta)B_{\tilde{n}_2} \tag{3.23}$$

is of rank 2 for generic θ .

As an application, let $\mathcal{C}$ be an irreducible component of $\mathcal{L}_2$ (see (3.11) for definition) containing a point λ spanned by $\tilde{n}_0$ and $\tilde{n}_1$, for which $r_\lambda = 2$. Let S^6 be the standard unit sphere in $\tilde{n}_0^\perp$, the Euclidean space spanned by $\tilde{n}_1, \cdots, \tilde{n}_7$, and let $\mathcal{C}_0$ be the connected component of the (real) variety

$$\mathcal{C}_0 := \{\tilde{n} \in S^6 \subset \tilde{n}_0^\perp : \textit{oriented 2-plane spanned by } (\tilde{n}_0, \tilde{n}) \in \mathcal{C}\}$$

containing $\tilde{n}_1$. The circle

$$\gamma(\theta) := \cos(\theta)\tilde{n}_1 + \sin(\theta)\tilde{n}_2$$

spans the so-called **tangent cone** *$\mathcal{T}$ of $\mathcal{C}_0$ at $\tilde{n}_1$ in S^6, as $\tilde{n}_2$ by our construction above are tangents to all possible analytic curves through $\tilde{n}_1$ in $\mathcal{C}_0$. By (3.23), for a generic $\tilde{n}$ on $\mathcal{T}$, the 2-plane spanned by $(\tilde{n}_0, \tilde{n})$ belongs to $\mathcal{L}_2$.*

Note, in particular, that when $\tilde{n}_1$ is a generic smooth point on $\mathcal{C}_0$, the tangent cone $\mathcal{T} \subset \mathcal{L}_2$ is just the standard unit sphere in the linear space spanned by $\tilde{n}_1$ and the tangent space of $\mathcal{C}_0$ at $\tilde{n}_1$.

r-nullity turns out to be crucial for understanding the structure of an isoparametric hypersurface when its multiplicity pair is $(7, 8)$. As an immediate application, let us sharpen the lower bound in (3.14).

Lemma 3.2. *Let $(m_+, m_-) = (7,8)$. Fix λ_0 in an irreducible component $\mathcal{C}$ of $\mathcal{L}_j$. Let λ_0 be spanned by the frame $(\tilde{n}_0, \tilde{n}_1)$ and extend it to the normal basis $\tilde{n}_0, \tilde{n}_1, \tilde{n}_2, \cdots, \tilde{n}_7$, with $S_{\tilde{0}}$ and $S_{\tilde{1}}$ normalized as in Convention* 3.1, (2.4), *and* (2.5). *Suppose no normal basis elements $\tilde{n}_2, \cdots, \tilde{n}_7$ are j-null. Then over $\mathcal{C}$ we have*

$$m_- \geq 2k - j - c_j, \tag{3.24}$$

where c_j is the codimension of $\mathcal{C}$ in $\mathcal{Q}_{k-1}$ (see (3.15)*).*

Proof. $r_\lambda = j$ for each $\lambda \in \mathcal{L}_j$ by definition. By (3.8) and (3.16), $p_{\tilde{0}} = 0$ cuts $\mathscr{S}_{\lambda_0}$ in the variety

$$\{(X_1, X_2, Y_1, Y_2, Z_1, Z_2)\},$$

where

$$X_1 = (x_1, \cdots, x_{8-j}), \qquad X_2 = (x_{9-j}, \cdots, x_8),$$
$$Z_1 = (z_1, \cdots, z_{7-j}), \qquad Z_2 = (z_{8-j}, \cdots, z_7),$$

satisfy (j is r in (3.16))

$$\sum_{\alpha=1}^{8-j} x_\alpha^2 = 0, \tag{3.25}$$

$X_1 = \pm\sqrt{-1}Y_1$, $X_2 = -Y_2$, and Z_2 depends linearly on X_2 (and vice versa). Since no bases are j-null, we may assume some $p_{\tilde{l}}, l \geq 2$, does not annihilate $\mathscr{S}_{\lambda_0}$, so that Lemma 3.1 implies that in the expression (see (3.17))

$$p_l = \sum_{\alpha=1,p=1}^{8-j,7-j} (S^l_{\alpha p} + \pm\sqrt{-1}T^l_{\alpha p}) x_\alpha z_p + \text{ other terms}, \tag{3.26}$$

the displayed sum is nontrivial. (3.25) and (3.26) imply that $p_{\tilde{0}} = p_{\tilde{l}} = 0$ cuts down one more dimension in $\mathscr{S}_{\lambda_0}$, which remains true for a generic $\lambda \in \mathcal{C}$, so that the lower bound in (3.14) is reduced further by 1 to yield (3.24) for a generic λ.

On the other hand, since those nongeneric $\lambda \in \mathcal{C}$ constitute a subvariety of codimension at least 1, the lower bound in (3.24) still holds ture over this subvariety. q.e.d.

4. Constraints on $1, 2, 3$-nullity

Lemma 4.1. *Let $(m_+, m_-) = (7,8)$. Away from points of Condition A on M_+, no element of a normal basis can be* 1*-null.*

Proof. Suppose n_j is 1-null for some $j \geq 2$. Set (B_j, C_j) to be of the form

$$B_j = \begin{pmatrix} 0 & d_j \\ b_j & c_j \end{pmatrix}, \quad C_j = \begin{pmatrix} 0 & g_j \\ e_j & f_j \end{pmatrix}, \tag{4.1}$$

for some real numbers c_j and f_j. We show $d_j = g_j = c_j = f_j = 0$.

Indeed, with

$$A_j = \begin{pmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{pmatrix}, \quad A_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad B_1 = C_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}, \tag{4.2}$$

one derives (we suppress the index for notational ease) that $c = f = \delta = 0$, and

$$\begin{aligned} &\beta = -\sqrt{2}g,\ \gamma = -\sqrt{2}d^{tr},\ \alpha\gamma^{tr} = \alpha\beta = 0,\ |d| = |g|, \\ &\alpha\alpha^{tr} + \beta\beta^{tr} + 2dd^{tr} = I,\ b = e,\ |\gamma|^2 + 2|b|^2 = 1. \end{aligned} \tag{4.3}$$

Indeed, (3.19a) through (3.19f) let us obtain the first, second, and sixth of the equations, together with $c = f$ and $\delta = 0$, while from (3.19e) there follows $c = f = 0$. On the other hand, (2.3a) results in $\alpha\gamma^{tr} = 0$, the fifth, and the seventh of the equations. A symmetric argument invoking (2.3b) replaces γ^{tr}, d, b by β, g, e, respectively. Lastly, (2.3f) implies the fourth equation.

Suppose $d \neq 0$. By a basis change we may assume

$$d = (t, 0, 0, \cdots, 0)^{tr}$$

for some positive number t. The skew-symmetry of α and the second and third identities of (4.3) ensure that the first row and column of α are zero.

If the first entry of g is zero, by a basis change we may assume

$$g = (0, s, 0, 0, \cdots, 0)$$

for some positive s, so that the third identity implies that the first two rows and columns of α are zero. Ignoring these trivial rows and columns of α, we see that the remainder of it, denoted $\tilde{\alpha}$ of size 5 by 5, is skew-symmetric, orthogonal and satisfies

$$\tilde{\alpha}^2 = -Id.$$

That is, $\mathbb{R}^5$ is acted on by $\tilde{\alpha}$ as a Clifford C_1-module, so that 5 is divisible by 2, a contradiction. Therefore, the first entry of g is not zero. In particular, the fifth identity implies that all the other entries of g are zero. Meanwhile, the first, fourth and fifth identities derive $|d| = 1/2 = t$, so that

$$\gamma = (-\sqrt{2}/2, 0, 0, \cdots, 0), \quad \beta = \pm\gamma, \quad d = \pm g, \quad |b| = 1/2.$$

But now (2.3h) for $i = j$ gives

$$b^{tr}(d^{tr}g + g^{tr}d) = 0. \tag{4.4}$$

Consequently, we obtain $d = g = 0$, which is contradictory.

With $d = g = c = f = 0$, B_j is of rank $r = 1$ in the form as in (2.6), which is impossible by item (1) of Corollary 2.1 since $r = 1$.

q.e.d.

Lemma 4.2. *Let $(m_+, m_-) = (7, 8)$. Away from points of Condition A on M_+, notation is as in* (4.1) *and* (4.2) *with the spectral data change that now*

$$A_1 = \begin{pmatrix} I & 0 \\ 0 & \Delta \end{pmatrix}, \quad B_1 = C_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}, \quad \sigma = sId, \quad \Delta = \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}$$

with $t = \sqrt{1 - 2s^2}$.

- **(1):** *If a normalized basis $n_0, n_1, \cdots, n_7$ is such that the generic rank of the linear combinations of $B_2, \cdots, B_7 \geq 5$, then it cannot be 2-null.*
- **(2):** *If the basis elements n_2, n_3, n_4 are 2-null, and generic linear combinations of B_1, B_2, B_3, B_4 are of rank ≤ 2, then $r_\lambda \leq 2$ for any λ in the 3-quadric of oriented 2-planes of $\mathbb{R}^5$ linearly spanned by $n_0, \cdots, n_4$.*

Proof. To prove item(1), let $n_0, n_1, \cdots, n_7$ be a 2-null basis. Let $n = a_2 n_2 + \cdots + a_7 n_7$ be a unit normal vector. Then

$$B_n := \sum_{j=2}^{7} a_j B_j = \begin{pmatrix} 0 & \sum_{j=2}^7 a_j d_j \\ \sum_{j=2}^7 a_j b_j & \sum_{j=2}^7 a_j c_j \end{pmatrix} := \begin{pmatrix} 0 & d_n \\ b_n & c_n \end{pmatrix}.$$

It follows that the rank of B_n is ≤ 4 by a dimension count, a contradiction.

To prove item (2), supposing first that

(I) all $b_1, \cdots, b_4$ are zero.

We employ Remark 2.1 to calculate r_λ.

Since $B_1, \cdots, B_4$ and $C_1, \cdots, C_4$ are of the form

$$B_i := \begin{pmatrix} 0 & d_i \\ 0 & c_i \end{pmatrix}, \quad C_i = \begin{pmatrix} 0 & g_i \\ 0 & f_i \end{pmatrix},$$

where c_i, f_i are of size r-by-r ($r = 2$; we are doing a general argument), a linear combination of $S_c := c_0 S_0 + \cdots + c_4 S_4$ assumes the form

$$S_c := \begin{pmatrix} c_0 I & A_c & B_c \\ A_c^{tr} & -c_0 I & C_c \\ B_c^{tr} & C_c^{tr} & 0 \end{pmatrix}, \tag{4.5}$$

where A_c, B_c, C_c are the linear combinations of A_i, B_i, C_i with coefficients $c_i, 1 \leq i \leq 4$. It follows that the vector

$$\mathbb{R}^8 \oplus \mathbb{R}^8 \oplus \mathbb{R}^7 \ni \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}, \quad z = \begin{pmatrix} u \\ 0 \end{pmatrix},$$

where u is of size $(m_+ - r)$-by-1, belongs to the kernel of S_c for all c. Therefore, the kernels of any two S_c and $S_{c'}$ intersect in a space of dimension at least $m_+ - r$, so that by Remark 2.1

$$r_\lambda \leq m_+ - (m_+ - r) = r,$$

where λ is the 2-plane spanned by the two vectors

$$c_0 n_0 + \cdots + c_4 n_4, \quad c_0' n_0 + \cdots + c_4' n_4.$$

Consequently, generic r_λ for $\lambda \in \mathcal{Q}_3$ is r, where $\mathcal{Q}_3$ is the set of oriented 2-planes in the Euclidean space spanned by $n_0, \cdots, n_4$.

Otherwise, we may assume

(II) b_2 is nonzero.

(2.3f) for $i = 1, j \geq 2$ gives

$$b_j = e_j, \quad c_j - f_j = -(c_j - f_j)^{tr}.$$

Meanwhile, the same identity for $i = j \geq 2$ derives

$$b_j^{tr}(c_j - f_j) = 0, \quad d_j^{tr} d_j + c_j^{tr} c_j = g_j^{tr} g_j + f_j^{tr} f_j. \tag{4.6}$$

Since $c_2 - f_2$ is 2-by-2 and skew-symmetric, it follows by (4.6) that $c_2 = f_2$. Since a generic linear combination of b_2, b_3, b_4 can be renamed to be b_2, it furthermore follows that

$$c_j = f_j, \quad 2 \leq j \leq 4,$$

and so by the second identity of (4.6), we obtain

$$d_j^{tr} d_j = g_j^{tr} g_j, \quad 2 \leq j \leq 4. \tag{4.7}$$

Now, (2.3e) for $i \geq 2, j = 1$ asserts

$$\sigma \delta_i \sigma - \sigma \Delta (c_i - f_i) \quad \text{is skew-symmetric,}$$

so that $\sigma \delta_i \sigma$ is skew-symmetric as $c_i = f_i$. Thus we deduce

$$\delta_i = \begin{pmatrix} 0 & a_i \\ -a_i & 0 \end{pmatrix}$$

for some number a_i. This imposes one linear constraint. Hence we may assume $\delta_2 = 0$ in the linear span of B_2, B_3, B_4. (Note that with this frame change b_2 need not be nonzero anymore.)

Next, (2.3a) and (2.3b) for $i \geq 2, j = 1$ result in

$$\delta_i^{tr} \Delta - \Delta \delta_i = -2s(f_i + f_i^{tr}) = -2s(c_i + c_i^{tr}) = -\delta_i \Delta + \Delta \delta_i^{tr}. \tag{4.8}$$

It follows that

$$c_i = f_i = \begin{pmatrix} 0 & p_i \\ -p_i & 0 \end{pmatrix}, \quad 2 \leq i \leq 3, \tag{4.9}$$

for some numbers p_i, which imposes another linear constraint. We may therefore assume

$$c_2 = f_2 = \delta_2 = 0 \tag{4.10}$$

in the span of B_2, B_3, B_4. With (4.10), (2.3a) and (2.3b) for $i = j = 2$ give

$$\begin{aligned}
&\alpha_2\alpha_2^{tr}+\beta_2\beta_2^{tr}+2d_2d_2^{tr}=Id, \quad \alpha_2^{tr}\alpha_2+\gamma_2^{tr}\gamma_2+2g_2g_2^{tr}=Id,\\
&\gamma_2\gamma_2^{tr}+2b_2b_2^{tr}=Id, \quad \beta_2^{tr}\beta_2+2b_2b_2^{tr}=Id,\\
&\alpha_2\gamma_2^{tr}=0, \quad \alpha_2^{tr}\beta_2=0,\\
&\alpha_2=-\alpha_2^{tr},
\end{aligned} \tag{4.11}$$

where we remark that the last identity comes from setting $i=2, j=1$ in (2.3a).

(IIa) d_2 **cannot be of rank 2.**

Suppose d_2 is of rank 2. Then $b_2=0$; or else B_2 written as in (4.1) would be of rank 3 by row reduction. Now, since generic linear combination of b_2, b_3, b_4 is nonzero, we may assume $b_3 \neq 0$. It follows that

$$\cos(\theta)B_2+\sin(\theta)B_3=\begin{pmatrix} 0 & \cos(\theta)d_2+\sin(\theta)d_3 \\ \sin(\theta)b_3 & \sin(\theta)c_3 \end{pmatrix}$$

is of rank at least 3 for a small angle θ, because d_2 is of rank 2 and b_3 is of rank at least 1. Therefore, the generic linear combination of B_2, B_3, B_4 is of rank ≥ 3, a contradiction. We next observe that

(IIb) d_2 **cannot be of rank 0.**

This is because otherwise from (4.7) we obtain

$$d_2=g_2=0.$$

Now (3.19a) and (3.19b) are just

$$\beta_j=(d_j\Delta-g_j)\sigma^{-1}, \quad \gamma_j^{tr}=-(d_j+g_j\Delta)\sigma^{-1}, \quad j\geq 2; \tag{4.12}$$

in particular,

$$\beta_2=\gamma_2=0.$$

With (4.11), we arrive at

$$A_2=\begin{pmatrix} \alpha_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad B_2=\begin{pmatrix} 0 & 0 \\ b_2 & 0 \end{pmatrix}, \quad \alpha_2\alpha_2^{tr}=Id, \quad b_2b_2^{tr}=Id.$$

Now, (2.3a) and (2.3b) for $i=2, j=3,4$ give

$$\alpha_2\alpha_j=-\alpha_j\alpha_2, \quad \alpha_2\beta_j=0, \quad \alpha_2\gamma_j^{tr}=0, \quad j=3,4,$$

from which there follows $\beta_j=\gamma_j^{tr}=0$, so that (4.12) implies $d_j=g_j=0, j=3,4$, and so (2.3a) and (2.3b) derive

$$\alpha_i\alpha_j=-\alpha_j\alpha_i, \quad \alpha_i^2=-Id, \quad 2\leq i\neq j\leq 4.$$

However, this says $\alpha_2, \alpha_3, \alpha_4$ induce a Clifford C_3-action on $\mathbb{R}^6$, so that 4 divides 6, a contradiction. Therefore,

(IIc) d_2 **must be of rank 1.**

Observe that

(IIc1) α_2 **cannot be of rank 6**; otherwise, the fifth and sixth identities of (4.11) force $\beta_2 = \gamma_2 = 0$ and so (4.12) gives $d_2 = g_2 = 0$, which is impossible.

Being skew-symmetric, α_2 must then be of even rank ≤ 4. We may thus write

$$\alpha_2 = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 \\ \beta \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \gamma \end{pmatrix}, \tag{4.13}$$

where α is of rank 0, 2, 4. β is of size 6-by-2, 4-by-2, 2-by-2, and γ is of size 2-by-6, 3-by-4, 2-by-2, respectively.

(IIc2) α **cannot be of rank 0.** Suppose the contrary. β and γ^{tr} are both of size 6-by-2. In particular, d_2 and g_2 are of the same form as β_2 and γ_2, respectively. The first identity of (4.11) gives

$$\beta_2\beta_2^{tr} = I - 2d_2d_2^{tr}.$$

Since the 6-by-6 $d_2d_2^{tr}$ is of rank at most 2 (because d_2 is of size 6-by-2), it has eigenvalue 0 counted at least four times, so that $I - 2d_2d_2^{tr}$ has eigenvalue 1 counted at least 4 times and so its rank is at least 4, which contradicts the fact that $\beta_2\beta_2^{tr}$ is of rank at most 2 (because β_2 is of size 6-by-2).

(IIc3) α **cannot be of rank 2.** Suppose the contrary. We remark that in general any A_j and B_j can be brought to the normalized form of A_1 and B_1 as in (2.4) and (2.5). That is, with an appropriate basis change we have

$$B_j = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_j \end{pmatrix}, \quad A_j = \begin{pmatrix} I & 0 \\ 0 & \Delta_j \end{pmatrix}, \tag{4.14}$$

where σ_j is diagonal and the nonzero part of Δ_j is skew-symmetric in the same form as σ and Δ in (2.4) and (2.5). In particular, suppose σ_j is of size 3-by-3, then $\Delta_j\Delta_j^{tr}$ has a zero eigenvalue so that one of the eigenvalues of σ_j is $1/\sqrt{2}$.

Now, as a consequence of (4.13) and (4.12), we obtain

$$d_2 = \begin{pmatrix} 0 \\ d \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 \\ g \end{pmatrix},$$

where β, γ^{tr}, d and g are all of size 4-by-2. The first two identities of (4.11) give

$$\beta\beta^{tr} + 2dd^{tr} = Id, \quad \gamma^{tr}\gamma + 2gg^{tr} = Id, \tag{4.15}$$

from which it follows that the 4-by-4 $\beta\beta^{tr}$, being of rank ≤ 2, has eigenvalue 0 counted at least twice, so that dd^{tr} has eigenvalue $1/\sqrt{2}$ counted at least twice. That is, the 2-by-2 $d^{tr}d$ has eigenvalue $1/\sqrt{2}$ counted exactly twice, so that d_2 is of rank 2. But d_2 is of rank 1, a contradiction. So now,

(IIc4) α **must be of rank** 4**.**

We may assume

$$d = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} u & v \\ w & z \end{pmatrix}. \tag{4.16}$$

It is important to remark that d can be put in the above diagonal form without changing the values of the normalized A_1 and B_1 in (2.4) and (2.5). In fact, we can first perform a row operation to bring d to an upper triangular form without changing σ in $B_1 = C_1$. Now due to the fact that $\sigma = sI$, we can then perform a row operation to bring d to the diagonal form. By doing so, we do have to conduct a row operation also on the rows of σ to let σ continue to be sI.

We employ (4.7) to conclude that

$$v = z = 0, \qquad u^2 + w^2 = p^2.$$

Moreover, (4.12) gives

$$\beta = s^{-1}\begin{pmatrix} -u & pt \\ -w & 0 \end{pmatrix}, \quad \gamma^{tr} = s^{-1}\begin{pmatrix} -p & -tu \\ 0 & -tw \end{pmatrix}. \tag{4.17}$$

Substituting them into (4.15) we obtain

$$u = 0, \quad w^2 = p^2 = s^2. \tag{4.18}$$

We leave it as a simple exercise to conclude the following

Sublemma 4.1. $c_2 = c_3 = c_4 = 0$. *Moreover, either*

$$b_i = \begin{pmatrix} 0 & 0 & 0 & 0 & w_i \\ 0 & 0 & 0 & 0 & z_i \end{pmatrix}, \quad d_i = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ y_i & -x_i \\ 0 & 0 \end{pmatrix},$$

where $w_i = \frac{\sqrt{(1-t^2)}}{s} x_i$ *and, moreover,* $z_i = \frac{\sqrt{(1-t^2)}}{s} y_i$ *if* $x_i \neq 0$, *for all* $2 \leq i \leq 4$, *or*

$$b_i = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ t_{i1} & t_{i2} & t_{i3} & t_{i4} & t_{i5} \end{pmatrix}, \quad d_i = \begin{pmatrix} u_{i1} & 0 \\ u_{i2} & 0 \\ u_{i3} & 0 \\ u_{i4} & 0 \\ u_{i5} & 0 \end{pmatrix}$$

for all $2 \leq i \leq 4$.

Proof. (sketch) We know $c_2 = 0$ by (4.10). By the third identity of (4.11), (4.16), (4.17), and (4.18), we obtain

$$b_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{(1-t^2)/2)} \end{pmatrix}, \quad d_2 = \begin{pmatrix} 0 & 0 \\ p & 0 \\ 0 & 0 \end{pmatrix}, \quad t \neq 1 \text{ as } s \neq 0,$$

with an appropriate column operation on b_2 (note that b_2 is of size 2-by-5 and d_2 of size 6-by-2). The sublemma follows by the fact that any linear combination of $B_1, \cdots, B_4$ is of rank ≤ 2 and so all its 3-by-3 minors are zero while invoking (4.9). q.e.d.

To finish the proof of the lemma, we shall find the intersection of the kernel spaces of two neighboring S_c and $S_{c'}$ given in (4.5) for generic choices of c and c'. Let $(x, y, z)^{tr}, x, y \in \mathbb{R}^8, z \in \mathbb{R}^7$, be in the kernel space of S_c, which amounts to the following

$$(4.19)\quad c_0 x + A_c y + B_c z = 0, \quad A_c^{tr} x - c_0 y + C_c z = 0, \quad B_c^{tr} x + C_c^{tr} y = 0.$$

Since the choice of c is generic, B_c is of rank 2, so that we can change frame in which (2.4) and (2.5) hold for B_c with

$$B_c = C_c = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_c \end{pmatrix}, \quad A_c = \begin{pmatrix} I & 0 \\ 0 & \Delta_c \end{pmatrix}.$$

The point is that then the third identity of (4.19) implies that if we decompose x, y, z, relative to the new frame, into

$$x = (X_1, X_2), \quad y = (Y_1, Y_2), \quad z = (Z_1, Z_2), \quad X_2, Y_2, Z_2 \in \mathbb{R}^2,$$

then $X_2 = -Y_2$ in the space V_c perpendicular to the kernel of B_c^{tr} (V_c is the image of B_c). Meanwhile, the first and second identities result in

$$Z_2 = -c_0 X_2 + \Delta_c X_2, \quad X_1 = Y_1 = 0,$$

so that the kernel of S_c is parametrized by Z_1 in the kernel of B_c and X_2 in the image of B_c ($=V_c$), which is 7-dimensional.

In both cases of the above sublemma, the two generic c and c' introduce a 1-dimensional reduction to the 5-dimensional kernel of B_c, whereas the image of B_c ($= V_c$) retains a common space for the kernels of S_c and $S_{c'}$. In fact, in the former case,

$$\text{kernel}\,(B_c^{tr}) = \mathbb{R}^5 \oplus L_c \subset \mathbb{R}^5 \oplus \mathbb{R}^3,$$

where L_c is a line. Therefore, V_c is a 2-plane perpendicular to L_c in $\mathbb{R}^3$, so that V_c and $V_{c'}$ intersect in a line in $\mathbb{R}^3$. In the latter case, $V_c \cap V_{c'}$ is the last (8^{th}) coordinate line of x.

In any event, the kernels of S_c and $S_{c'}$ intersect in a space of dimension 5, 4 dimensions from the intersection of the kernels of B_c and $B_{c'}$ and 1 dimension from $V_c \cap V_{c'}$. Thus, generically $r_\lambda = 7 - 5 = 2$.

q.e.d.

Lemma 4.3. *Let $(m_+, m_-) = (7, 8)$. Away from points of Condition A on M_+, let $n_0, \cdots, n_7$ be a normal basis such that the frame (n_0, n_1) is normalized with the given spectral data (σ, Δ) as in* (2.4) *and* (2.5)*. Assume σ is of size* 3-by-3 *and the generic rank of linear combinations of $B_1, \cdots, B_7$ is ≥ 5.*

(1): *If $\sigma \neq I/\sqrt{2}$, then the normal basis cannot be* 3*-null.*

(2): *Suppose n_2, n_3, n_4 are 3-null, and moreover, suppose the spectral data of all linear combinations of B_1 through B_4 are $(\sigma, \Delta) = (I/\sqrt{2}, 0)$. Then $b_2 = b_3 = b_4 = 0$ if all linear combinations of B_1 through B_4 are of rank ≤ 3. In particular, under the same condition, $r_\lambda \leq 3$ for any λ in the 3-quadric of oriented 2-planes of $\mathbb{R}^5$ linearly spanned by $n_0, \cdots n_4$.*

Proof. First note that the 3-by-3 matrices σ in $B_1 = C_1$ and Δ in A_1 are now

$$\sigma := \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & sId \end{pmatrix},\ s \neq 0;\ \Delta := \begin{pmatrix} 0 & 0 \\ 0 & tJ \end{pmatrix},\ J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ t = \sqrt{1-2s^2},$$

with A_j, B_j, C_j and the associated notation given in (4.1), (4.2). Since σ is of size 3-by-3, the skew-symmetric Δ must have a zero eigenvalue, which accounts for the eigenvalue $1/\sqrt{2}$ for σ.

Item (1).

$t \neq 0$ in this case. Suppose the normal basis $n_0, n_1, n_2, \cdots, n_7$ is 3-null.

By (3.19c), we have

$$b_j = e_j, \quad \sigma(c_j - f_j) = -(c_j - f_j)^{tr}\sigma, \quad 2 \leq j \leq 7. \tag{4.20}$$

The second identity of (4.20) gives

$$c_j - f_j = \begin{pmatrix} 0 & v \\ -v^{tr}/s\sqrt{2} & w \end{pmatrix}, \tag{4.21}$$

where w is 2-by-2 skew-symmetric. On the other hand, (2.3f) for $i = j \geq 2$ results in

$$b_j^{tr}(c_j - f_j) = 0, \quad d_j^{tr} d_j + c_j^{tr} c_j = g_j^{tr} g_j + f_j^{tr} f_j. \tag{4.22}$$

Since the b-matrices are of rank at least 2 generically for the generic B_n matrices to be of rank ≥ 5, we see from (4.21) and (4.22) that the $c - f$ matrices are zero generically and hence are zero identically, so that now

$$c_j = f_j, \quad 2 \leq j \leq 7; \tag{4.23}$$

with $h = 0$ and σ diagonal in (3.19f), there now follows

$$\delta_j \text{ is skew-symmetric}, \quad 2 \leq j \leq 7. \tag{4.24}$$

Meanwhile, the $(2,2)$-block of (2.3g) asserts

$$(-\Delta^2 + \sigma^2)c_j + 2c_j\sigma^2 + \sigma c_j^{tr}\sigma + \sigma^2 c_j + \sigma c_j^{tr}\sigma = c_j,$$

which comes down to

$$c_j\sigma = -\sigma c_j^{tr}, \quad 2 \leq j \leq 7,$$

which gives that c_j is of the form

$$c_j = \begin{pmatrix} 0 & c_{j1} \\ -c_{j1}^{tr}/s\sqrt{2} & c_{j2} \end{pmatrix}, \quad c_{j2} = -c_{j2}^{tr}, \quad 2 \leq j \leq 7,$$

where c_{j2} is 2-by-2.

Since the matrix form of $c_j, 2 \leq j \leq 7$, imposes three linear constraints, we may thus assume without loss of generality that

$$c_2 = f_2 = 0. \tag{4.25}$$

(2.3b) for $i = 2, j = 1$ then derives

$$\delta_2^{tr}\Delta - \Delta\delta_2 = -2(f_2 + f_2^{tr})\sigma = 0,$$

from which there follows, on account of (4.24) and $t \neq 0$,

$$\delta_2 = 0; \tag{4.26}$$

in particular, (4.11) holds true again.

Now the second identity of (4.22) and (4.23) imply

$$d_2^{tr} d_2 = g_2^{tr} g_2, \tag{4.27}$$

and moreover (2.3h) gives

$$b_2^{tr}(\beta_2^{tr} d_2 + \gamma_2 g_2) = 0,$$

which, when incorporated with (4.27) and (4.12), arrives at

$$b_2^{tr}\sigma^{-1}(d_2^{tr} g_2 + g_2^{tr} d_2) = 0. \tag{4.28}$$

Now, since the 5-by-5 α_2 is skew-symmetric, its rank is either 0, 2, or 4, so that β_2 and γ_2 being in the kernel of α_2 imply that we can assume

$$\alpha_2 = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 \\ \beta \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \gamma \end{pmatrix},$$

where α is of rank 0, 2, or 4 of the same square size, β is of size 5-by-3, 3-by-3, or 1-by-3, and γ is of size 3-by-5, 3-by-3, or 3-by-1, respectively.

(I) α cannot be zero.

Assume $\alpha = 0$. Now, since

$$B_2 B_2^{tr} = \begin{pmatrix} b_2 b_2^{tr} & 0 \\ 0 & d_2 d_2^{tr} \end{pmatrix}$$

is of rank at most 6 (because both b_2 and d_2 are of rank at most 3) , we see

$$A_2 A_2^{tr} = \begin{pmatrix} \beta_2 \beta_2^{tr} & 0 \\ 0 & \gamma_2 \gamma_2^{tr} \end{pmatrix}$$

has eigenvalue 1 counted at least twice, which implies by the third and fourth identities of (4.11) that $b_2 b_2^{tr}$ has eigenvalue 0 counted at least once so that, in particular, $b_2 b_2^{tr}$ is of rank at most 2 and hence $B_2 B_2^{tr}$ is of rank at most 5 and so in fact $A_2 A_2^{tr}$ has eigenvalue 1 counted at least three times. Thus, either $\beta_2 \beta_2^{tr}$ or $\gamma_2 \gamma_2^{tr}$ has eigenvalue 1 counted at least twice, so that $b_2 b_2^{tr}$ has eigenvalue 0 counted at least twice and so b_2 is of rank at most 1 and $B_2^{tr}B_2$ is of rank at most 4. This forces $A_2A_2^{tr}$ to have eigenvalue 1 counted at least four times; we conclude, by

$$\gamma_2\gamma_2^{tr} = \beta_2^{tr}\beta_2, \tag{4.29}$$

a consequence of the third and fourth identities of (4.11), that each of $\beta_2^{tr}\beta_2$ and $\gamma_2\gamma_2^{tr}$ has some eigenvalue $1-2\epsilon^2, \epsilon \leq 1/2$, counted once and eigenvalue 1 counted twice, whereas $d_2^{tr}d_2$ (and $g_2^{tr}g_2$) has the eigenvalue ϵ^2 counted once and eigenvalue $1/2$ counted twice and $b_2b_2^{tr}$ is of rank at most 1 with eigenvalue ϵ^2 counted once and eigenvalue 0 counted twice.

By performing a row operation without changing A_1, B_1, C_1, we may assume the 5-by-3 d_2 is of the form

$$d_2 = \begin{pmatrix} d \\ 0 \end{pmatrix}. \tag{4.30}$$

Write the 5-by-3 β_2 as

$$\beta_2 = \begin{pmatrix} \theta \\ \mu \end{pmatrix},$$

where θ is of size 3-by-3. The first identity of (4.11), with $\alpha_2 = 0$, gives

$$\theta\mu^{tr} = 0, \quad \mu\mu^{tr} = I. \tag{4.31}$$

Meanwhile, by (4.12)

$$g_2 = \begin{pmatrix} d\Delta - \theta\sigma \\ -\mu\sigma \end{pmatrix}, \quad \gamma_2^{tr} = \begin{pmatrix} -d\sigma^{-1}(I - \Delta^{tr}\Delta) + \theta\Delta \\ \mu\Delta \end{pmatrix}. \tag{4.32}$$

To facilitate the following calculations, let us observe that θ is at most 1-dimensional by (4.31), so that by performing row operations we may assume

$$\theta = \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad d = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ 0 & d_{32} & d_{33} \end{pmatrix}. \tag{4.33}$$

Set

$$|\theta|^2 := \theta_{11}^2 + \theta_{12}^2 + \theta_{13}^2.$$

Suppose $|\theta|^2 \neq 0$. Then (4.31) implies that

$$M := \begin{pmatrix} \theta_{11}/|\theta| & \theta_{12}/|\theta| & \theta_{13}/|\theta| \\ u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \end{pmatrix}$$

is an orthogonal matrix, where

$$\mu := \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \end{pmatrix}.$$

In particular, by the fact that all columns of M are mutually orthogonal unit vectors, we conclude that

$$\beta_2^{tr}\beta_2 = Id + (1 - \frac{1}{|\theta|^2}) \begin{pmatrix} \theta_{11} \\ \theta_{12} \\ \theta_{13} \end{pmatrix} \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} \end{pmatrix}. \tag{4.34}$$

We record from (4.32) the following identity

$$\begin{aligned} \gamma_2\gamma_2^{tr} &= (-d\sigma^{-1}(I - \Delta^{tr}\Delta) + \theta\Delta)^{tr}(-d\sigma^{-1}(I - \Delta^{tr}\Delta) + \theta\Delta) \\ &\quad + (\mu\Delta)^{tr}(\mu\Delta), \end{aligned} \tag{4.35}$$

whose right hand side calculates, if we write

$$M := \begin{pmatrix} -\sqrt{2}d_{11} & -2sd_{12} - \theta_{13}t & -2sd_{13} + \theta_{12}t \\ -\sqrt{2}d_{21} & -2sd_{22} & -2sd_{23} \\ 0 & -2sd_{32} & -2sd_{33} \end{pmatrix},$$

the equation

$$\gamma_2\gamma_2^{tr} = M^{tr}M + \begin{pmatrix} 0 & -u_{13}t & u_{12}t \\ 0 & -u_{23}t & u_{22}t \end{pmatrix}^{tr} \begin{pmatrix} 0 & -u_{13}t & u_{12}t \\ 0 & -u_{23}t & u_{22}t \end{pmatrix}. \tag{4.36}$$

Equating (4.34) and (4.36), we find in the $(2,2)$-, $(3,3)$-, $(1,2)$-, $(1,3)$-, and $(2,3)$-entries that there are terms linear in t, which are, respectively,

$$4std_{12}\theta_{13}, \quad 4std_{13}\theta_{12}, \quad d_{11}\theta_{13}t, \quad d_{11}\theta_{12}t, \quad -2std_{12}\theta_{12} + 2std_{13}\theta_{13}.$$

All other terms are quadratic in t, so that changing t to $-t$ we obtain

$$d_{12}\theta_{13} = d_{13}\theta_{12} = d_{11}\theta_{12} = d_{11}\theta_{13} = -d_{12}\theta_{12} + d_{13}\theta_{13} = 0. \tag{4.37}$$

As a consequence, $\theta_{12} = \theta_{13} = 0$ if $d_{11} \neq 0$, for which we have

$$\theta\Delta = 0. \tag{4.38}$$

Otherwise, $d_{11} = 0$. If $d_{12} \neq 0$, then $d_{13} = 0$, or else once more $\theta_{12} = \theta_{13} = 0$. But then the last identity of (4.37) implies that again $\theta_{12} = \theta_{13} = 0$. In any event, we conclude that $\theta_{12} = \theta_{13} = 0$ so that (4.38) is true, unless

$$d_{11} = d_{12} = d_{13} = 0,$$

in which case the identity

$$2dd^{tr} + \theta\theta^{tr} = Id$$

returns

$$|\theta|^2 = 1, \quad dd^{tr} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}. \tag{4.39}$$

By (4.34) we have

$$\gamma_2\gamma_2^{tr} = \beta_2^{tr}\beta_2 = Id, \tag{4.40}$$

so that (4.35) now calculates the $(1,1)$-entry to yield

$$d_{21} = \pm 1/\sqrt{2}.$$

Meanwhile, the $(1,2)$- and $(1,3)$- entries result in

$$d_{22} = d_{23} = 0.$$

Use (4.40) to calculate the $(2,3)$-entry of $\gamma_2\gamma_2^{tr}$ to obtain

$$-(\theta_{12}\theta_{13} + u_{12}u_{13} + u_{22}u_{23})t^2 + 4s^2 d_{32}d_{33} = 0,$$

so that

$$d_{32}d_{33} = 0$$

since the matrix M above with $|\theta| = 1$ is orthogonal.

Suppose $d_{33} = 0$. Then the $(3,3)$-entry gives

$$(\theta_{12}^2 + u_{12}^2 + u_{22}^2)t^2 = 1,$$

so that $t^2 = 1$ since M with $|\theta| = 1$ is orthogonal, which contradicts $s > 0$.

Suppose $d_{32} = 0$. The $(2,2)$-entry gives

$$(\theta_{13}^2 + u_{13}^2 + u_{23}^2)t^2 = 1,$$

which again contradicts $s > 0$.

In conclusion, (4.38) holds true when $\theta \neq 0$, while it is trivially true if $\theta = 0$. As a consequence, we conclude

$$\mu = \begin{pmatrix} 0 & U \end{pmatrix}, \quad UU^{tr} = I, \quad \theta = \begin{pmatrix} \tau & 0 \end{pmatrix} \tag{4.41}$$

where U is of size 2-by-2 and τ is of size 3-by-1, when we invoke the second identity of (4.31); moreover, with (4.41) in hands we can now assume

$$d_{21} = d_{32} = 0 \tag{4.42}$$

in d by performing row operations without affecting (4.41).

Now that $\theta\Delta = 0$, from which γ_2 is simplified to facilitate the calculation of (4.35) to derive, by (4.29),

$$(I+\Delta^2)^{tr}\sigma^{-1}d^{tr}d\sigma^{-1}(I+\Delta^2)+\Delta^{tr}\mu^{tr}\mu\Delta = \theta^{tr}\theta+\mu^{tr}\mu = \mathrm{diag}(|\tau|^2, 1, 1)$$

whose right hand side gives the eigenvalues of $\beta_2^{tr}\beta_2$, which, as we mentioned above, are 1 counted twice and $|\tau|^2 = 1 - 2\epsilon^2$; when we invoke (4.41), the equality simplifies to

$$d^{tr}d = \begin{pmatrix} (1-2\epsilon^2)/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}. \tag{4.43}$$

Therefore, since the eigenvalues of $d_2^{tr}d_2$ are ϵ^2 counted once and $1/2$ counted twice, it implies

$$\epsilon^2 = (1-2\epsilon^2)/2, \quad \text{so } \epsilon^2 = 1/4.$$

On the other hand, $d^{tr}d$ can be calculated by (4.33) and (4.42) to obtain

$$d_{12} = d_{13} = d_{23} = 0, \quad d_{11}^2 = 1/4,\ d_{22}^2 = d_{33}^2 = 1/2.$$

The fourth identity of (4.11) now gives

$$b_2 b_2^{tr} = (I - \beta_2^{tr}\beta_2)/2 = \begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \epsilon^2 = 1/4,$$

while in (4.28)

$$d_2^{tr} g_2 + g_2^{tr} d_2 = \begin{pmatrix} -\sqrt{2}p\tau_1 & -q\tau_2/\sqrt{2} & -r\tau_3/\sqrt{2} \\ -q\tau_2/\sqrt{2} & 0 & (q^2-r^2)t \\ -r\tau_3/\sqrt{2} & (q^2-r^2)t & 0 \end{pmatrix}, \quad \tau = (\tau_1.\tau_2, \tau_3)^{tr}.$$

In particular, (4.28) forces $\tau = 0$, which is a contradiction as $|\tau|^2 = 1 - 2\epsilon^2 = 1/2$.

(II) α cannot be of rank 2.

Assume α is of rank 2 so that β and γ are both of size 3-by-3; by (4.12) d_2 and g_2 are of the same form as β_2 and γ_2, respectively. Write

$$d_2 = \begin{pmatrix} 0 \\ X \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 \\ Y \end{pmatrix}, \tag{4.44}$$

where X and Y are made up of 3-by-1 column vectors X_1, X_2, X_3 and Y_1, Y_2, Y_3, respectively.

(IIa) If d_2 is of rank 3, then (4.27) implies that there is a 3-by-3 orthogonal matrix T such that

$$TX_i = Y_i, \quad 1 \le i \le 3.$$

(IIa1) If b_2 is of rank 3, (4.28) gives

$$X_i \cdot TX_j = -X_j \cdot TX_i, \quad 1 \le i, j \le 3,$$

where $\cdot$ denotes the standard inner product. Consequently, T is skew-symmetric and orthogonal. This is impossible as $\det(T) = 0$ now.

(IIa2) If b_2 is of rank ≤ 2, then $b_2 b_2^{tr}$ has an eigenvalue 0, so that by the fourth identity of (4.11) $\beta_2^{tr}\beta_2$ has an eigenvalue 1. By the first identity of (4.11), this forces XX^{tr} to have an eigenvalue 0, so that d_2 is not of rank 3, a contradiction.

(IIb) If d_2 is of rank ≤ 2, then note that since the lower right 2-by-2 block of σ is a multiple of the identity matrix, we can perform column operations between the last two columns of X without changing A_1, B_1 and C_1, though we cannot perform column operations to interchange the first and the remaining two columns if we want to retain the values of A_1, B_1 and C_1, for reason that $s \ne 1/\sqrt{2}$.

By performing a row operation without changing A_1, B_1, C_1, we may assume the 3-by-3 X takes the form

$$X = \begin{pmatrix} d \\ 0 \end{pmatrix}, \quad d = \begin{pmatrix} x & y & z \\ 0 & w & u \end{pmatrix}, \tag{4.45}$$

where X and Y are given in (4.44). For notational consistence, we set

$$\beta_2 = \begin{pmatrix} 0 \\ \beta \end{pmatrix}, \quad \beta = \begin{pmatrix} \theta \\ \mu \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \gamma \end{pmatrix}, \quad Y = g,$$

where β, Y are of size 3-by-3 and θ is of size 2-by-3.

We have two cases to consider, where when $x = 0$ we may perform row and column operations to assume $y \neq 0$ and $w = 0$.

(IIb1) When x is nonzero, we change our notation from (4.45) to denote d by

$$d = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ 0 & d_{22} & d_{23} \end{pmatrix},$$

for more notational consistence; we will return to the notation in (4.45) later. Write

$$\theta := \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & \theta_{23} \end{pmatrix}, \quad \mu := \begin{pmatrix} u_{11} & u_{12} & u_{13} \end{pmatrix}, \quad \theta\mu^{tr} = 0, \quad \mu\mu^{tr} = 1. \tag{4.46}$$

The right hand side of (4.35), if we write

$$T := \begin{pmatrix} -\sqrt{2}d_{11} & -2sd_{12} - \theta_{13}t & -2sd_{13} + \theta_{12}t \\ 0 & -2sd_{22} - \theta_{23}t & -2sd_{23} + \theta_{22}t \end{pmatrix},$$

now reads

$$\gamma_2\gamma_2^{tr} = T^{tr}T + \begin{pmatrix} 0 & -u_{13}t & u_{12}t \end{pmatrix}^{tr} \begin{pmatrix} 0 & -u_{13}t & u_{12}t \end{pmatrix}. \tag{4.47}$$

Similar to the arguments leading to (4.37), we derive by (4.47) the constraints

$$\begin{aligned} &d_{11}\theta_{12} = d_{11}\theta_{13} = 0,\ d_{12}\theta_{13} + d_{22}\theta_{23} = 0,\ d_{13}\theta_{12} + d_{23}\theta_{22} = 0, \\ &-d_{12}\theta_{12} + d_{13}\theta_{13} - d_{22}\theta_{22} + d_{23}\theta_{23} = 0. \end{aligned} \tag{4.48}$$

Suppose $d_{11} \neq 0$. We first treat the case when $\theta \neq 0$. Then

$$\theta_{12} = \theta_{13} = 0, \quad \theta = \begin{pmatrix} \theta_{11} & 0 & 0 \\ \theta_{21} & \theta_{22} & \theta_{23} \end{pmatrix}.$$

If either $d_{22} \neq 0$ or $d_{23} \neq 0$, then $\theta_{22} = \theta_{23} = 0$, from which there follows that μ and θ are of the form

$$\mu = (0, 0, 1), \quad \theta = \begin{pmatrix} p & q & 0 \\ r & l & 0 \end{pmatrix}. \tag{4.49}$$

by the third identity in (4.46) since $\theta \neq 0$. We may now assume

$$d_{22} = d_{23} = 0.$$

So now since the only nonzero row of d is the first one,

$$d_2 = \begin{pmatrix} 0 \\ X \end{pmatrix}, \quad X = \begin{pmatrix} d \\ 0 \end{pmatrix}$$

is also of rank 1. There follows that g_2 is of rank 1 by (4.27). Meanwhile, the third identity in (4.46) implies

$$\theta_{11} u_{11} = 0, \quad \mu = \begin{pmatrix} u_{11} & u_{12} & u_{13} \end{pmatrix},$$

so that

$$u_{11} = 0$$

if $\theta_{11} \neq 0$, which gives again the condition (4.49). This is because it says that the first component of the vector μ is zero, and since $|\mu| = 1$ by the last identity in (4.46), we may perform column operations on the last two columns to assume that in fact

$$\mu = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix},$$

and hence the third identity in (4.46) gives

$$\theta = \begin{pmatrix} \theta_{11} & \theta_{12} & 0 \\ \theta_{21} & \theta_{22} & 0 \end{pmatrix},$$

giving (4.49). Therefore, we may now assume

$$\theta_{11} = 0.$$

Hence both d and θ can have only one nonzero row; explicitly

$$d = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ 0 & 0 & 0 \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 & 0 & 0 \\ \theta_{21} & \theta_{22} & \theta_{23} \end{pmatrix}.$$

In particular, (4.32) calculates, in the notation of (4.44),

$$g_2 = \begin{pmatrix} 0 \\ Y \end{pmatrix}, \quad Y = \begin{pmatrix} d\Delta - \theta\sigma \\ -\mu\sigma \end{pmatrix} = \begin{pmatrix} 0 & -d_{13}t & d_{12}t \\ -\theta_{21}/\sqrt{2} & -s\theta_{22} & -s\theta_{23} \\ -\mu_{11}/\sqrt{2} & -s\mu_{12} & -s\mu_{13} \end{pmatrix},$$

which is of rank 1, since g_2 and d_2 are of the same rank by (4.27) and

$$d_2 = \begin{pmatrix} 0 \\ X \end{pmatrix}, \quad X = \begin{pmatrix} d \\ 0 \end{pmatrix},$$

where X, of size 3-by-3, is of rank 1; thus, $\mu_{11} = 0$ if either $d_{12} \neq 0$ or $d_{13} \neq 0$, which gives us the condition (4.49) as explained above. As a result, we may now assume

$$d_{12} = d_{13} = 0. \tag{4.50}$$

Now with

$$d = \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

(4.27) asserts

$$Y^{tr}Y = d^{tr}d = \begin{pmatrix} d_{11}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

from which the $(2,2)$- and $(3,3)$-entry of $Y^{tr}Y$ give

$$s^2(\theta_{22}^2 + u_{12}^2) = 0, \quad s^2(\theta_{23}^2 + \mu_{13}^2) = 0,$$

that is

$$\theta_{22} = \theta_{23} = u_{12} = u_{13} = 0.$$

Now that

$$\theta = \begin{pmatrix} 0 & 0 & 0 \\ \theta_{21} & 0 & 0 \end{pmatrix},$$

the third identity in (4.46) implies

$$\theta_{21}\mu_{11} = 0,$$

or, equivalently,

$$\mu_{11} = 0$$

since we assume $\theta \neq 0$ so that the only nonzero component $\theta_{21} \neq 0$; this is the condition (4.49) again.

Continue to assume $d_{11} \neq 0$. Let us handle the situation when $\theta = 0$, for which

$$\beta_2 = \begin{pmatrix} 0 \\ \beta \end{pmatrix}, \quad \beta = \begin{pmatrix} \theta \\ \mu \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ u_{11} & u_{12} & u_{13} \end{pmatrix}$$

is of rank 1, so that γ_2^{tr} will be of rank 1 by (4.27). Now by (4.32),

$$\gamma_2^{tr} = \begin{pmatrix} 0 \\ \gamma^{tr} \end{pmatrix},$$

$$\gamma^{tr} = \begin{pmatrix} -d\sigma^{-1}(Id - \Delta^{tr}\Delta) + \theta\Delta \\ \mu\Delta \end{pmatrix} = \begin{pmatrix} -\sqrt{2}d_{11} & -2sd_{12} & -2sd_{13} \\ 0 & -2sd_{22} & -2sd_{23} \\ 0 & -u_{13}t & u_{12}t \end{pmatrix},$$

whose rank being 1 implies

$$d_{22} = d_{23} = u_{12} = u_{13} = 0$$

since $d_{11} \neq 0$. But then

$$\beta = \begin{pmatrix} \theta \\ \mu \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \pm 1 & 0 & 0 \end{pmatrix}, \quad d = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ 0 & 0 & 0 \end{pmatrix},$$

which contradicts

$$2XX^{tr} + \beta\beta^{tr} = Id,$$

an identity extracted from the lower 3-by-3 block of the first identity of (4.11).

In conclusion, the condition (4.49) holds true if $d_{11} \neq 0$.

Next, we consider the situation when $d_{11} = 0$ (already $d_{21} = 0$). Since the first column of d is zero, by row and column operations we my assume

$$d_{12} \neq 0, \quad d_{22} = 0.$$

Now observe that with $d_{11} = d_{21} = 0$, the first column and row of $\gamma_2\gamma_2^{tr}$ given in (4.47) are zero. On the other hand, $\beta^{tr}\beta$ must have the same property by (4.29) since

$$\beta^{tr}\beta = \gamma\gamma^{tr};$$

it follows by computing its $(1,1)$-entry, where

$$\beta = \begin{pmatrix} \theta \\ \mu \end{pmatrix} = \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & \theta_{23} \\ u_{11} & u_{12} & u_{13} \end{pmatrix},$$

that

$$\theta_{11}^2 + \theta_{21}^2 + u_{11}^2 = 0,$$

so that

$$u_{11} = \theta_{11} = \theta_{21} = 0;$$

returning to the notation in (4.45), this reads $x = w = 0$, where (4.49) remains true with $p = r = 0$.

With these remarks out of the way, (4.32) gives

$$g = \begin{pmatrix} -p/\sqrt{2} & -zt - sq & yt \\ -r/\sqrt{2} & -ut - sl & wt \\ 0 & 0 & -s \end{pmatrix}, \quad \gamma^{tr} = \begin{pmatrix} -\sqrt{2}x & -2sy & -2sz + tq \\ 0 & -2sw & -2su + tl \\ 0 & -t & 0 \end{pmatrix}.$$

We calculate to see

$$\beta^{tr}\beta = \begin{pmatrix} p^2 + r^2 & pq + rl & 0 \\ pq + rl & q^2 + l^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\gamma\gamma^{tr} =$$

$$\begin{pmatrix} 2x^2 & \cdot & \cdot \\ 2\sqrt{2}sxy & 4s^2y^2 + 4s^2w^2 + t^2 & \cdot \\ -\sqrt{2}x(-2sz + tq) & \begin{matrix} -2sy(-2sz + tq) - \\ 2sw(-2su + tl) \end{matrix} & \begin{matrix} (-2sz + tq)^2 + \\ (-2su + tl)^2 \end{matrix} \end{pmatrix},$$

where we dot certain entries to indicate that the matrix is symmetric.

By (4.29), if $x \neq 0$,

$$-2sz + tq = 0, \quad -2sw(-2su + tl) = 0, \quad (-2su + tl)^2 = 1, \quad \text{so } w = 0;$$

on the other hand, if $x = w = 0$ and $y \neq 0$, we obtain

$$-2sy(-2sz + tq) = 0, \quad (-2su + tl)^2 = 1, \quad \text{so } -2sz + tq = 0.$$

In any event,

$$-2sz + tq = 0, \quad (-2su + tl)^2 = 1, \quad w = 0. \tag{4.51}$$

With these refined data, we observe that the $(2,2)$ entry of $\gamma^{tr}\gamma$ is 1, and thus we can employ the second identity of (4.11) to conclude that the $(2,2)$-entry of gg^{tr} is zero, i.e.,

$$r = 0, \quad -tu - sl = 0, \tag{4.52}$$

(IIb2) In the case when x is zero, we compare the $(2,3)$ entries of (4.7) to conclude

$$yz = (-zt - sq)ty, \quad \text{so } \ z = -zt^2 - stq$$

which, when incorporated with (4.51), arrives at

$$z = -zt^2 - stq = -zt^2 - 2s^2 z = -(t^2 + 2s^2)z = -z, \quad \text{so } \ z = q = 0.$$

But then the $(2,2)$ entry of (4.7) gives

$$y^2 = (zt + sq)^2 = 0,$$

a contradiction.

(IIb3) Therefore, $x \neq 0$ is the only possibility, where $w = r = 0$ as verified above. We now have the simplified data

$$\begin{aligned} d &= \begin{pmatrix} x & y & z \\ 0 & 0 & u \end{pmatrix}, \qquad g = \begin{pmatrix} -p/\sqrt{2} & -zt - sq & yt \\ 0 & 0 & 0 \\ 0 & 0 & -s \end{pmatrix}, \\ \beta &= \begin{pmatrix} p & q & 0 \\ 0 & l & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \gamma^{tr} = \begin{pmatrix} -\sqrt{2}x & -2sy & 0 \\ 0 & 0 & \pm 1 \\ 0 & -t & 0 \end{pmatrix}. \end{aligned} \tag{4.53}$$

Accordingly, (4.29) simplifies to

$$2x^2 = p^2, \quad pq = 2\sqrt{2}sxy, \quad q^2 + l^2 = 4s^2y^2 + t^2.$$

Since $x \neq 0$, we incorporate (4.51) and (4.52) to solve these equations to obtain

$$p = \pm\sqrt{2}x, \quad q = \pm 2sy, \quad z = \pm ty, \quad l^2 = t^2, \tag{4.54}$$

where the first three equalities share the same sign. We then employ the last equality of (4.54) and the second equality of (4.52) to see

$$l = \pm t, \quad u = \mp s,$$

which means that l and u must differ by a sign. However, since l appears in the second column and u appears in the third, we can certainly change the sign of the basis vector to change the sign of one of l and u without affecting the other, while keeping the values of A_1, B_1 and C_1, to arrange that l and u have the same sign. This is a contradiction.

(III) α cannot be of rank 4.

Suppose α is of rank 4. Then β and γ^{tr} are 1-by-3. It follows by (4.12) that X and Y are 1-by-3. Write

$$X := (a, b, c), \quad \beta := (p, q, r),$$

where X is given in (4.44). Then (4.12) gives, as above,

$$\begin{aligned} Y &= (-p/\sqrt{2}, -tc - sq, tb - sr),\\ \gamma^{tr} &= (-\sqrt{2}a, (t^2-1)b/s - tr, (t^2-1)c/s + tq). \end{aligned}$$

Meanwhile, $X^{tr}X = Y^{tr}Y$ and $\beta^{tr}\beta = \gamma\gamma^{tr}$ derive as above

$$\begin{aligned} &a = \pm(-p/\sqrt{2}), \quad b = \pm(-tc - sq), \quad c = \pm(tb - sr),\\ &p = \pm(-\sqrt{2}a), \quad q = \pm((t^2-1)b/s - tr), \quad r = \pm((t^2-1)c/s + tq), \end{aligned}$$

where the three equations in each of the two triples share the same sign. It follows that, by solving the linear system with the unknowns a, b, c, p, q, r, we obtain

$$b = c = q = r = 0, \tag{4.55}$$

since $s \neq 1/\sqrt{2}$. Then, by the third identity of (4.11) we obtain

$$2b_2b_2^{tr} = \begin{pmatrix} 1-2a^2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix},$$

so that we see from the $(1,1)$-entry of (4.28) that

$$(1-2a^2)a^2 = 0.$$

If $a = 0$, then $X = Y = 0$, or rather $d_2 = g_2 = 0$, so that by (4.12) $\beta_2 = \gamma_2 = 0$, which contradicts the first identity of (4.11). Hence, $2a^2 = 1$. But then the first identity of (4.11) results in

$$1 = p^2 + q^2 + r^2 + 2a^2 + 2b^2 + 2c^2 = p^2 + 2a^2 = p^2 + 1;$$

we conclude that $p = 0$, or rather $\beta = 0$, so that $\gamma = 0$ by $\beta^{tr}\beta = \gamma\gamma^{tr}$, and so (4.12) gives $d_2 = g_2 = 0$, a contradiction again. We are done with item (1).

Item (2).

We assume that a generic linear combination of B_1 through B_4 is of rank 3. Then the linear combination

$$B(\theta) := \cos(\theta)B_1 + \sin(\theta)B_2 = \begin{pmatrix} 0 & \sin(\theta)d_2\\ \sin(\theta)b_2 & \cos(\theta)I + \sin(\theta)c_2 \end{pmatrix}$$

is of rank 3 for a generic θ, with (B_1, C_1) and (B_2, C_2) given in (2.4) and (4.1), where now

$$\sigma = I/\sqrt{2}, \quad \Delta = 0;$$

in particular, (2.3a), (2.3b), and (2.3e) for $i=2, j=1$ assert
(4.56)
$$c_2=-(c_2)^{tr},\ \ f_2=-(f_2)^{tr},\ \ \delta_2=-(\delta_2)^{tr},\ \ \gamma_2^{tr}=-\sqrt{2}d_2,\ \ \beta_2=-\sqrt{2}g_2,$$

The kernel of the 8-by-7 $B(\theta)$ is of dimension 4 for a generic θ. Setting $(x,y)^{tr}$ for a kernel vector of $B(\theta)$, where x is of size 1-by-4 and y is of size 1-by-3, we solve to see

$$\sin(\theta)d_2y=0,\quad \sin(\theta)b_2x+(\cos(\theta)I/\sqrt{2}+\sin(\theta)c_2)y=0,$$

from which we derive

$$d_2(\cos(\theta)I/\sqrt{2}+\sin(\theta)c_2)^{-1}b_2x=0,\quad \forall x.$$

It follows that

$$0=d_2(\cos(\theta)I/\sqrt{2}+\sin(\theta)c_2)^{-1}b_2=\frac{\sqrt{2}}{\cos(\theta)}\sum_{k=0}^{\infty}(-1)^k d_2(c_2)^k b_2 z^k,$$

where we set $z=\sqrt{2}\tan(\theta)$, for a generic small θ, which is equivalent to

$$d_2(c_2)^k b_2=0,\quad k=0,1,2,3,\cdots \tag{4.57}$$

Likewise, by considering C_2 we obtain

$$g_2(f_2)^k b_2=0,\quad k=0,1,2,3,\cdots \tag{4.58}$$

(I) The case when d_2 is of rank 3.

Performing a row reduction on the matrix B_2, we can eliminate c_2 without changing b_2. It follows that $b_2=0$ because B_2 is of rank 3. But since a generic linear combination of d_2,d_3,d_4 is also of rank 3, we see a generic linear combination, and hence all linear combinations of b_2,b_3,b_4 are zero. We are done.

We may now assume that all linear combinations of d_2,d_3,d_4 (likewise, of g_2,g_3,g_4) are of rank at most 2.

(II) The case when generic linear combination of d_2,d_3,d_4 is of rank $=2$.

Assume d_2 is of rank 2.

(IIa) If c_2 is nonzero, performing row and column operations, without changing B_1,C_1, and A_1, we may assume

$$c_2=zJ,\quad J:=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix},\quad z\neq 0;$$

this is possible because the spectral data $(\sigma, \Delta) = (I/\sqrt{2}, 0)$ now. We then perform a column operation on the last two columns without changing A_1, B_1, C_1 and c_2, so that we may assume

$$d_2 = \begin{pmatrix} p & q & 0 \\ 0 & r & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34}, \end{pmatrix}. \tag{4.59}$$

from which (4.57) for $k = 0, 1$ with $z \neq 0$ results in

$$pb_{11} + qb_{21} = 0, \quad rb_{21} + ub_{31} = 0, \quad qb_{31} = 0, \quad -ub_{21} + rb_{31} = 0.$$

Generically, we may always assume $p \neq 0$ (by performing row and column operations if necessary). We solve to see that $b_2 = 0$ by the fact that one of r and u is nonzero for d_2 to have rank 2. Since the choice of n_2 is generic, this says that $b_2 = b_3 = b_4 = 0$ if generic combinations of c_2, c_3, c_4 are not zero. We are done.

(IIb) So now we may assume

$$c_2 = c_3 = c_4 = 0, \quad \text{and likewise} \quad f_2 = f_3 = f_4 = 0,$$

and a generic combination of b_2, b_3, b_4 is nonzero, which we may assume is b_2, without loss of generality.

The rank of g_2 is also 2, because (2.3f) for $i = j = 2$ reads

$$d_2^{tr} d_2 = g_2^{tr} g_2, \tag{4.60}$$

knowing that $c_2 = f_2 = 0$.

Setting $k = 0$ in (4.57) and (4.58), we see that the column space of b_2 is identical with the 1-dimensional kernel space of d_2 and of g_2. We may thus assume b_2 is spanned by $(0, 0, 1)^{tr}$ and assume

(4.61)

$$d_2 = \begin{pmatrix} p & q & 0 \\ 0 & r & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} 0 & x & y \\ -x & 0 & w \\ -y & -w & 0 \end{pmatrix}.$$

(2.3a) applied to $i = j = 2$ gives

$$\gamma_2 \gamma_2^{tr} + \delta_2 \delta_2^{tr} + 2 b_2 b_2^{tr} = I; \tag{4.62}$$

with the fourth identity of (4.56) one compares the $(1, 3), (2, 3)$, and $(3, 3)$-entries to ensure

$$xy = xw = 0, \quad 2a^2 + y^2 + w^2 = 1.$$

(IIb0) If x is nonzero, then $y = w = 0$ and $a^2 = 1/2$, from which we see the nonzero 2-by-2 block d of d_2 satisfies

$$d^{tr} d = (1 - x^2) I/2, \quad \text{so} \quad q = 0, \quad p^2 = r^2 = (1 - x^2)/2,$$

incorporating the fourth identity of (4.56) and (4.62). However, since the spectral data, which are $(\sigma, \Delta) = (I/\sqrt{2}, 0)$ by assumption, of B_2 are the nonzero eigenvalues of $d_2^{tr} d_2$ and $b_2 b_2^{tr}$ in view of the fact that we can now derive

$$B_2 B_2^{tr} = \begin{pmatrix} d_2 d_2^{tr} & 0 \\ 0 & b_2 b_2^{tr} \end{pmatrix},$$

we therefore conclude that $(1-x^2)/2 = 1/2$, i.e., $x = 0$, a contradiction. We thus conclude that

(IIb1) x **must be zero.**

(IIb11) If either y **or** w **is nonzero,** we observe first that with $c_2 = 0$, (2.3a) for $i = j = 2$ implies

$$\alpha_2 \gamma_2^{tr} = -\beta_2 \delta_2^{tr}, \tag{4.63}$$

which says, by reading the third columns on both sides while invoking the fourth and fifth identities of (4.56), that the first two columns of g_2 are linearly dependent with coefficients y and w, whereas (4.60) asserts that the third column of g_2 is zero. This forces g_2 to be of rank ≤ 1, a contradiction. Consequently, it must be that

(IIb12) $x = y = w = 0$ **so that** $\delta_2 = 0$.

Now that $c_2 = f_2 = \delta_2 = 0$, observe further that since the spectral data is $(\sigma, \Delta) = (I/\sqrt{2}, 0)$, the general identity (4.35) asserts that

$$\gamma\gamma^{tr} = 2d^{tr}d,$$

If the ranks of α and d_2 are 2, since β is also of rank 2 by (4.29), we see that

$$A_2 A_2^{tr} = \begin{pmatrix} \alpha\alpha^{tr} & 0 & 0 \\ 0 & \beta\beta^{tr} & 0 \\ 0 & 0 & \gamma\gamma^{tr} \end{pmatrix}$$

gives that A_2 is of rank 6, so that the spectral data could not be $(\sigma, \Delta) = (I/\sqrt{2}, 0)$ because it would force the rank of A_2 to be $8 - 3 = 5$. This case does not occur.

On the other hand, the same proof as in item (1), III, in the case when the rank of α is 4 gets us all the way through to the linear system above (4.55), where our spectral data is now $(\sigma, \Delta) = (I/\sqrt{2}, 0)$. It is easily checked that

$$\beta\beta^{tr} = \gamma^{tr}\gamma = 1, \quad XX^{tr} = YY^{tr} = 1. \tag{4.64}$$

Now, $A_2^{tr} A_2$ is of rank 6 with eigenvalue 1 counted six times, four times from α and once each from β and γ, and 0 counted twice, so that $B_2 B_2^{tr}$ is of rank 2 with eigenvalue $1/2$ counted twice. This again contradicts our spectral data assumption $(\sigma, \Delta) = (I/\sqrt{2}, 0)$. This case does not occur either.

(III) Generic linear combination of d_2, d_3, d_4 cannot be of rank < 2.

We first assume d_2 is the generic choice of rank 1. We may also assume that b_2 is the generic choice such that $b_2 \neq 0$; otherwise, generic linear combination of b_2, b_3, b_4 being zero implies that $b_2 = b_3 = b_4 = 0$, and we are done. We know by a symmetric reasoning that g_2 has rank ≤ 1.

(IIIa) Assume c_2 is nonzero. Notation as in (4.59), we remark that the setup in the preceding case is still valid with

$$r = u = 0$$

now. We manipulate essentially the same to yield that if $q \neq 0$, then $b_{31} = 0$ and $pb_{11} + qb_{21} = 0$, so that b_2 is of rank 1 as $b_2 \neq 0$. But then the matrix

$$B_2 = \begin{pmatrix} 0 & d_2 \\ b_2 & c_2 \end{pmatrix}$$

will be of rank 4, where the last row of c_2 (that of b_2 is 0) annihilates q and r of d_2 in a row operation; this is a contradiction. Hence, $q = 0$, from which it follows that $b_{1j} = 0$, i.e., the first row of b_2 is zero. Observe now we have

$$d_2 c_2 = 0, \quad c_2 = zJ,$$

so that we calculate

$$B_2 B_2^{tr} = \begin{pmatrix} d_2 d_2^{tr} & d_2 c_2^{tr} \\ c_2 d_2^{tr} & c_2 c_2^{tr} + b_2 b_2^{tr} \end{pmatrix} = \begin{pmatrix} d_2 d_2^{tr} & 0 \\ 0 & c_2 c_2^{tr} + b_2 b_2^{tr} \end{pmatrix};$$

therefore, the spectral data dictates that we have

$$c_2 c_2^{tr} + b_2 b_2^{tr} = \begin{pmatrix} 0 & 0 \\ 0 & z^2 I \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & bb^{tr} \end{pmatrix}, \quad b_2 := \begin{pmatrix} 0 \\ b \end{pmatrix}, \quad p^2 = 1/2,$$

where I of size 2-by-2, and b of size 2-by-3 satisfies

$$z^2 I + bb^{tr} = I/2. \tag{4.65}$$

Hence, the identity

$$\gamma_2 \gamma_2^{tr} + \delta_2 \delta_2^{tr} + 2(b_2 b_2^{tr} + c_2 c_2^{tr}) = I,$$

obtained by (2.3a) for $i = j = 2$, translates into

$$\gamma_2 \gamma_2^{tr} + \delta_2 \delta_2^{tr} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

As a consequence, $\delta_2 \delta_2^{tr} = 0$ because $p^2 = 1/2$ and $\gamma_2 = -\sqrt{2} d_2$. That is,

$$\delta_2 = 0;$$

with this (2.3a) now gives

$$\alpha_2 \gamma_2^{tr} = -d_2 c_2^{tr} = 0,$$

which implies that the first column (and hence the first row) of $\alpha_2 = 0$. Incorporating this into $p^2 = 1/2$ and

$$\alpha_2\alpha_2^{tr} + 2g_2g_2^{tr} + 2d_2d_2^{tr} = I \tag{4.66}$$

obtained by (2.3a), we conclude that the first column and row of $g_2g_2^{tr}$ are zero. That is, the first row of g_2 is zero; moreover, comparing the (1.1)-entries and knowing $p^2 = 1$, we see that the first column and row of α_2 are zero since it is skew-symmetric. Thus we can perform column and row operations, respecting A_1, B_1, C_1, d_2 and c_2, such that

$$g_2 = \begin{pmatrix} 0 & 0 & 0 \\ \theta & \epsilon & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, since

$$2g_2^{tr}g_2 + 2b_2b_2^{tr} + 2f_2f_2^{tr} = I \tag{4.67}$$

obtained by (2.3b) for $i = j = 2$ with $\delta_2 = 0$, we find that the $(1,3)$- and $(2,3)$-entries of $f_2f_2^{tr}$ are zero. That is,

$$eg = eh = 0, \quad f_2 := \begin{pmatrix} 0 & e & l \\ -e & 0 & h \\ -l & -h & 0 \end{pmatrix}.$$

If $e \neq 0$, then $l = h = 0$, so that inserting the first equality of (4.65) into (4.67) to compare the $(3,3)$-entries we obtain $z = 0$, a contradiction to $c_2 \neq 0$. Thus $e = 0$. We derive, by (2.3b) for $i = j = 2$,

$$\alpha_2^{tr}g_2 = -\sqrt{2}g_2f_2^{tr},$$

where the $(2,3)$-entry of the right hand side is a linear combination of the $(2,1)$- and $(2,2)$-entries of g_2 with coefficients l and h and all other entries are zero, whereas the $(2,3)$-entry of the the left hand side is zero. It follows that

$$g_2f_2^{tr} = 0 = \alpha_2^{tr}g_2,$$

from which we conclude that the second, in addition to the first, column and row of α_2 are zero. Thus, (2.3b) derives

$$\alpha\alpha^{tr} = I, \quad \alpha_2 = \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix},$$

because both $d_2d_2^{tr}$ and $g_2g_2^{tr}$ are nontrivial only at the upper left 2-by-2 block, where α is of size 3-by-3 and skew-symmetric, which is absurd. As a result,

(IIIb) c_2 **must be zero**.

Now that $c_2 = 0$, we employ (2.3f), which gives

$$d_2^{tr}d_2 = g_2^{tr}g_2 + f_2^{tr}f_2,$$

to observe that g_2 cannot be of rank 0, or else the left hand side is of rank 1 whereas the right hand side is of rank either 0 or 2. That is, g_2 must be of rank 1 as well, so that exactly the same parallel argument, replacing d_2 by g_2, establishes $f_2 = 0$. With now $c_2 = f_2 = \delta_2 = 0$, the same arguments in the paragraph containing (4.64) results in a contradiction. This case does not occur.

Lastly, it is impossible that both $d_2 = g_2 = 0$; for otherwise $\beta_2 = \gamma_2 = 0$. (2.3a) then asserts that the 5-by-5 skew-symmetric α_2 satisfies $\alpha_2\alpha_2^{tr} = I$, which is absurd.

q.e.d.

5. M_+ is generically 4-null

Lemma 5.1. *Let $(m_+, m_-) = (7, 8)$. Away from points of Condition A on M_+, suppose*

$$\sup_{\lambda \in \mathcal{Q}_6} r_\lambda \geq 5.$$

Then there is a choice of $p_0, \cdots, p_5$ for the codimension 2 estimate (3.14) to go through. In particular, $V_0, \cdots, V_5$ are irreducible and $p_0, \cdots, p_6$ form a regular sequence.

Proof. Recall the *a priori* codimension 2 estimate (3.14), which is

$$8 = m_- \geq 2k + 1 - j - c_j, \tag{5.1}$$

where $\mathcal{L}_j$ and c_j are defined in (3.11) and (3.15). It verifies that the codimension 2 estimate goes through for $k \leq 3$ and any choice of $p_0, \cdots, p_3$.

For $k = 4$, the estimate goes through for $j \geq 1$. However, since M_+ away from points of Condition A is not 0-null, item (2) of Corollary 2.1 implies that for $k = 4$, $\mathcal{L}_0$ is of codimension at least 1 in $\mathcal{Q}_3$ (i.e., $c_0 \geq 1$), because by the corollary there must be a $\lambda \in \mathcal{Q}_3$ for which $r_\lambda \neq 0$; therefore, the codimension 2 estimate goes through, for any choice of $p_0 \cdots p_4$. In particular, $V_0, \cdots, V_4$ are irreducible and any choice of $p_0, \cdots, p_5$ form a regular sequence.

For $k = 5$, we pick $p_0, \cdots, p_5$ such that

$$\sup_{\lambda \in \mathcal{Q}_4} r_\lambda \geq 5. \tag{5.2}$$

Note that (5.1), which is now

$$8 \geq 11 - j - c_j, \tag{5.3}$$

implies that the codimension 2 estimate automatically goes through for $j \geq 3$.

In general, for $j \leq 4$, $\mathcal{L}_j \subset \mathcal{Q}_4$ is not generic by (5.2), so that $c_j \geq 1$. Hence, (5.3) also takes care of the codimension 2 estimate for $j = 2$.

Moreover, since by Lemma 4.1, M_+ is not j-null for $j=1$, the refined codimension 2 estimate (3.24), which is

$$8 = m_- \geq 2k - j - c_j, \tag{5.4}$$

is satisfied for $j = 1, k = 5$ and $c_j \geq 1$; so, the codimension 2 estimate goes through for $j = 1$ as well.

For $j = 0$, (5.4) is ineffective as its right hand side is 9 with $c_j = 1$; we need to cut down one more dimension from its right hand side. That is, more fundamentally we must effectively cut $\mathscr{S}_\lambda, \lambda \in \mathcal{L}_0$, for generic $\lambda \in \mathcal{L}_0$.

Note, however, notation as in Convention 3.1, since $r_\lambda = 0$ for $\lambda \in \mathcal{L}_0$, we have $B_{\tilde{1}} = C_{\tilde{1}} = 0$ and $A_{\tilde{1}} = Id$ in (2.5) for $S_{\tilde{1}}$. It follows that $p_{\tilde{0}} = 0$ cuts $\mathscr{S}_\lambda$ in the variety

$$\{(x, \pm\sqrt{-1}x, z) : \sum_{\alpha=1}^{8} (x_\alpha)^2 = 0\}. \tag{5.5}$$

We may assume $(B_{\tilde{2}}, C_{\tilde{2}})$ of $S_{\tilde{2}}$ is nonzero away from points of Condition A. Since z is a free variable in (5.5), $p_{\tilde{2}} = 0$ cuts $\mathscr{S}_\lambda$ to result in the equation with nontrivial z-terms:

$$0 = p_{\tilde{2}} = \sum_{\alpha=1,p=1}^{8,7} (S^2_{\alpha p} \pm \sqrt{-1}T^2_{\alpha p}) x_\alpha z_p. \tag{5.6}$$

Hence by Lemma .7 in Appendix I, $p_{\tilde{2}} = 0$ introduces a nontrivial cut into $\mathscr{S}_\lambda$ to reduce the dimension estimate by 1, and more importantly the variety $\mathscr{F}_2$ cut out by $p_{\tilde{0}} = p_{\tilde{2}} = 0$ in (5.5) and (5.6) is irreducible. Indeed, we have seen before that this gives (5.4).

To cut one more dimension, we remark that one of the pairs $(B_{\tilde{3}}, C_{\tilde{3}})$, $(B_{\tilde{4}}, C_{\tilde{4}})$, and $(B_{\tilde{5}}, C_{\tilde{5}})$ is nonzero, to be in accordance with item (2) of Corollary 2.1. Hence we may assume none of them are zero by a generic rotation of the basis elements $n_{\tilde{3}}, n_{\tilde{4}}, n_{\tilde{5}}$; note that, with this, the variety $\mathscr{F}_i$ cut out by $p_{\tilde{0}} = p_{\tilde{i}} = 0, 3 \leq i \leq 5$, is also irreducible for the same reason as $\mathscr{F}_2$.

When $\mathscr{F}_2$ and $\mathscr{F}_j$ are distinct for some $j = 3,4,5$, $\mathscr{F}_2 \cap \mathscr{F}_j$ is of one dimension lower, i.e., $p_{\tilde{0}} = p_{\tilde{2}} = p_{\tilde{j}} = 0$ cuts down one more dimension in $\mathscr{S}_\lambda$ by Lemma .7 in Appendix I, so that the right hand side of (5.4) is dropped by 1 and so the codimension 2 estimate goes through.

We must then rule out the possibility that $\mathscr{F}_k, 2 \leq k \leq 5$, are all identical, or equivalently, that $p_{\tilde{j}}, j = 3,4,5$, restricted to $\mathscr{S}_\lambda$ are all constant multiples of $p_{\tilde{2}}$. That is,

$$S^i_{\alpha p} \pm \sqrt{-1}T^i_{\alpha p} = c_i(S^2_{\alpha p} \pm \sqrt{-1}T^2_{\alpha p}), \tag{5.7}$$

for some nonzero complex numbers $c_i, 3 \leq i \leq 5$.

Write

$$c_i = a_i + \sqrt{-1} b_i$$

for some real numbers a_i, b_i. Then we obtain

$$\begin{aligned} S^3_{\alpha p} &= a_3 S^2_{\alpha p} - b_3 T^2_{\alpha p}, \quad & T^3_{\alpha p} &= b_3 S^2_{\alpha p} + a_3 T^2_{\alpha p}, \\ S^4_{\alpha p} &= a_4 S^2_{\alpha p} - b_4 T^2_{\alpha p}, \quad & T^4_{\alpha p} &= b_4 S^2_{\alpha p} + a_4 T^2_{\alpha p}, \\ S^5_{\alpha p} &= a_5 S^2_{\alpha p} - b_5 T^2_{\alpha p}, \quad & T^5_{\alpha p} &= b_5 S^2_{\alpha p} + a_5 T^2_{\alpha p}. \end{aligned} \tag{5.8}$$

Choose a nonzero solution $(x, y, z), x^2 + y^2 + z^2 = 1$, to

$$a_3 x + a_4 y + a_5 z = 0, \quad b_3 x + b_4 y + b_5 z = 0. \tag{5.9}$$

Then it is easily seen that

$$x S^3_{\alpha p} + y S^4_{\alpha p} + z S^5_{\alpha p} = 0. \quad x T^3_{\alpha p} + y T^4_{\alpha p} + z T^5_{\alpha p} = 0. \tag{5.10}$$

That is, the shape operator $S_n := x S_{\tilde{3}} + y S_{\tilde{4}} + z S_{\tilde{5}}$ has the property that its B and C blocks are identically zero. So we may now assume the B and C blocks of $S_{\tilde{5}}$ are zero.

We may now ignore the above a_5 and b_5 in (5.9). Any nonzero solution (x, y) that solves the second equation of (5.9) implies that there is a real number c, namely, $c = a_3 x + a_4 y$, such that the B and C blocks of $S_n := x S_{\tilde{3}} + y S_{\tilde{4}}$ are c times of $B_{\tilde{2}}$ and $C_{\tilde{2}}$, respectively. But then $S_{n'}$, where $n' = (n_{\tilde{2}} - cn)/\sqrt{1 + c^2}$, has the property that the B and C blocks of $S_{n'}$ are zero. This means that we can now assume that the B and C blocks of $S_{\tilde{4}}$ are zero, with possible new $S_{\tilde{2}}$ and $S_{\tilde{3}}$ out of the Gram-Schmidt process. It follows that neither $(B_{\tilde{2}}, C_{\tilde{2}})$ nor $(B_{\tilde{3}}, C_{\tilde{3}})$ are zero to not to violate item (2) of Corollary 2.1.

We are now led to the conclusion that if an irreducible component $\mathcal{C}$ of $\mathcal{L}_0$ is such that the codimension 2 estimate is not true for all $\lambda \in \mathcal{C}$, then each $\lambda \in \mathcal{C}$ is contained in one and only one quadric $\mathcal{Q}_2 \subset \mathcal{C}$, which is the set of 2-planes in the 4-dimensional Euclidean space spanned by $\tilde{n}_0, \tilde{n}_1, \tilde{n}_4, \tilde{n}_5$ given in the preceding two paragraphs, where λ is the 2-plane spanned by $\tilde{n}_0, \tilde{n}_1$; in fact, this 4-dimensional linear space is characterized by that the shape operators S_n of all unit n in it share a common kernel (the Condition A for them). However, since $\mathcal{C}$ is of dimension 3 (recall $c_j = 1$ in (5.4) with $j = 0$), any two $\mathcal{Q}_2$ in $\mathcal{C}$ in $\mathcal{Q}_4$ will intersect in at least $2 + 2 - 3 = 1$ dimensional worth of points by a dimension count. It follows that for $\lambda \neq \lambda'$ in $\mathcal{C}$ belonging to their respective quadrics $Q_2 \neq Q_2'$, there is a third $\tau \in \mathcal{C}$ which lies in both Q_2 and Q_2', so that τ is contained in the join of these two quadrics; in other words, if τ is spanned by the orthonormal pair N_0 and N_1, then there are more than two orthonormal $N_j, j \geq 2$, in the join for which B_{N_j} is identically zero, which contradicts item (2) of Corollary 2.1. The contradiction implies that the codimension 2 estimate is true for at least one, and hence, for generic $\lambda \in \mathcal{C}$. q.e.d.

Under the assumption of Lemma 5.1, we further assume that the isoparametric hypersurface is *not* the one constructed by Ozeki and Takeuchi. Then by Lemma 5.1, away from points of Condition A on M_+, $p_0, \cdots, p_5$ form a regular sequence and $p_0 = \cdots = p_5 = 0$ carves out an irreducible variety V_5. It follows that $p_0, \cdots, p_6$ form a regular sequence for any choice of p_6 [**3**, Corollary 1, p. 6]. By (5.2), we also have

$$\sup_{\lambda \in \mathcal{Q}_5} r_\lambda \geq 5.$$

We know the codimension 2 estimate (5.4) can no longer go through for $k = 6$; or else $p_0, \cdots, p_7$ would be a regular sequence and the isoparametric hypersurface would be the one constructed by Ozeki and Takeuchi [**3**, Proposition 4, p. 11]. Let us understand how and where the codimension 2 estimate fails in this case.

For $k = 6$, when $(m_+, m_-) = (7, 8)$, we record that the *a priori* codimension 2 estimate (5.1) is now

$$8 = m_- \geq 13 - j - c_j. \tag{5.11}$$

CASE I. So clearly the codimension 2 estimate holds when $j \geq 4$ since $c_j \geq 1$ for $j \leq 4$.

CASE II. For $j = 3$, the codimension 2 estimate goes through as well as long as $c_j \geq 2$. So in the following we assume $c_j = 1$. We claim that the condition in Lemma 4.3, which is that the generic rank of the linear combinations of $B_1, \cdots, B_7$ is ≥ 5, is satisfied so that Lemma 3.2 allows us to employ the refined codimension 2 estimate (5.4), which is now,

$$8 = m_- \geq 12 - j - c_j, \tag{5.12}$$

to conclude that the codimension 2 estimate goes through with $j = 3$ and $c_j = 1$, as follows.

Lemma 5.2. *Let $\mathcal{C}$ be an irreducible component of $\mathcal{L}_j$. Suppose $\mathcal{C}$ is of codimension 1 in $\mathcal{Q}_5$ (i.e., $c_j = 1$). Then there is a $\lambda \in \mathcal{C}$, which is the 2-plane spanned by $\tilde{n}_0, \tilde{n}_1$, such that there is an $\tilde{n}_2$ perpendicular to $\tilde{n}_0, \tilde{n}_1$ for which $B_{\tilde{2}}$ is of rank at least 5.*

Proof. Let S^6 be the unit sphere in the linear space spanned by $n_0, \cdots, n_6$. Consider the incidence space

$$\mathcal{I} = \{(\tilde{n}, \lambda) \in S^6 \times \mathcal{C} :\ \tilde{n} \perp \tilde{n}_0, \tilde{n}_1;\ \lambda = \mathrm{span}(\tilde{n}_0, \tilde{n}_1)\}$$

with the projection π_1 and π_2 onto the first and second factors, respectively. $\mathcal{I}$ is (real) 12-dimensional because for each $\lambda = \mathrm{span}(\tilde{n}_0, \tilde{n}_1)$, the set $\pi_2^{-1}(\lambda)$ is the unit 4-sphere in the span of $\tilde{n}_2, \cdots, \tilde{n}_6$ perpendicular to $\tilde{n}_0, \tilde{n}_1$.

We show that π_1 is surjective. For each $\tilde{n}$ in the image of π_1, the set $\pi_1^{-1}(\tilde{n})$ consists of all $(\tilde{n}, \lambda), \lambda = \mathrm{span}(\tilde{n}_0, \tilde{n}_1) \in \mathcal{C}$, such that $\tilde{n} \perp \tilde{n}_0, \tilde{n}_1$; therefore, $\pi_1^{-1}(\tilde{n})$ is the intersection of $\mathcal{C}$ and the variety $\mathcal{G} \simeq \mathcal{Q}_4$ of oriented 2-planes in $\tilde{n}^\perp \simeq \mathbb{R}^6$ with $\tilde{n}$ in the span of $n_0, \cdots, n_6$ and so $\pi_1^{-1}(\tilde{n}) = \mathcal{G} \cap \mathcal{C}$ is (complex) 3-dimensional. As a result, $\pi_1(\mathcal{I})$ is (real) 6-dimensional contained in S^6 and so π_1 is surjective.

We can now pick a generic $\tilde{n} \in S^6$ whose associated $\mathcal{G} \cap \mathcal{C}$ recovers $\tilde{n}_0, \tilde{n}_1$ and designate this $\tilde{n}$ to be $\tilde{n}_2$. Then $B_{\tilde{2}}$ of $S_{\tilde{n}_2}$ assumes generic rank ≥ 5. q.e.d.

In view of the preceding lemma, if there is a $\lambda \in \mathcal{L}_3$ whose spectral data satisfy the condition in item (1) of Lemma 4.3, then the codimension 2 estimate goes through since the normal basis cannot be 3-null.

Otherwise, the spectral data of all $\lambda \in \mathcal{L}_3$ satisfy the condition in item (2) of Lemma 4.3. Now, pick a generic point $\lambda \in \mathcal{C}$ spanned by $n_{\tilde{0}}, n_{\tilde{1}}$. Let $S_{\tilde{0}}$ and $S_{\tilde{1}}$ be normalized as in (2.4) and (2.5) and extend them to $S_{\tilde{0}} \cdots, S_{\tilde{6}}$. Consider the $S^5 \subset \mathcal{Q}_5$ given by $[1 : \lambda_1 : \cdots : \lambda_6]$, where $\lambda_1, \cdots, \lambda_6$ are purely imaginary. Note that $\lambda = [1 : \sqrt{-1} : 0 : \cdots : 0]$ in $S^5 \cap \mathcal{C}$. Now,

$$\dim(S^5 \cap \mathcal{C}) \geq 5 + 8 - 10 = 3, \tag{5.13}$$

where 10 is the real dimension of $\mathcal{Q}_5$.

This dimension estimate implies that the closure Λ of the irreducible component of $S^5 \cap \mathcal{C}$ containing λ coincides with the unit 3-sphere of the span of $\tilde{n}_1, \tilde{n}_4, \tilde{n}_5, \tilde{n}_6$. This is because by the concluding paragraph of Remark 3.2, the closure of the irreducible component of $S^5 \cap \mathcal{C}$ containing $\tilde{n}_1$ is a sphere whose generic $\tilde{n}$ is 3-null. Thus, (5.13) implies that there are at least three such independent $\tilde{n}$, so that there are exactly three such independent $\tilde{n}$, namely, $\tilde{n}_4, \tilde{n}_5, \tilde{n}_6$ for $\tilde{n}_1, \tilde{n}_4, \tilde{n}_5, \tilde{n}_6$ to bound a 3-sphere, because $\tilde{n}_2$ is not 3-null since otherwise by item (2) of Lemma 4.3 the rank of $B_{\tilde{2}}$ would be 3, contradicting its being ≥ 5 as said in Lemma 5.2, and, consequently, $\tilde{n}_3$ is not 3-null either by virtue of (5.7). But then item (2) of Lemma 4.3 implies that all linear combinations of $B_{\tilde{4}}, B_{\tilde{5}}$, and $B_{\tilde{6}}$ are of the form in (4.1) with the b-block zero. It follows by item (2) of Lemma 4.3 that a generic point of the quadric $\mathcal{Q}_3$, defined to be the set of 2-planes in the span of $\tilde{n}_0, \tilde{n}_1, \tilde{n}_4, \tilde{n}_5, \tilde{n}_6$, is contained in $\mathcal{C}$, and moreover, this $\mathcal{Q}_3$ is the unique 3-quadric containing λ in the closure of $\mathcal{C}$ (because $\Lambda = S^3$).

But then, we can take a generic combination of $B_{\tilde{2}}, \cdots, B_{\tilde{6}}$, which is of rank 5, and call it $B_{2'}$ with normal direction n_2'. We then go through the same argument as above to conclude that we can come up with normal vectors n_4', n_5', n_6' such that $\tilde{n}_0, \tilde{n}_1, n_4', n_5', n_6'$ generate a $\mathcal{Q}_3'$ contained in the closure of $\mathcal{C}$ different from the above $\mathcal{Q}_3$, both containing λ. This contradicts the uniqueness of such $\mathcal{Q}_3$.

CASE III. For $j = 2$, Lemma 5.2 enforces item (1) of Lemma 4.2, so that Lemma 3.2 allows us to warrant the validity of (5.12), where the right hand side is ≤ 8; with $c_j \geq 2$ the codimension 2 estimate holds. Henceforth, we assume $c_j = 1$ and so $\mathcal{C} \subset \mathcal{Q}_5$ given in Lemma 5.2 is of (complex) dimension 4. The right hand side of (5.12) is 9; we need to cut down one more dimension for the codimension 2 estimate to go through. We spell out more details.

For $\lambda \in \mathcal{L}_2$, $p_{\tilde{0}} = p_{\tilde{1}} = 0$ cuts $\mathscr{S}_\lambda$ in the variety (see Lemma 3.2)

$$\{(X_1, X_2, Y_1, Y_2, Z_1, Z_2)\},$$

where $X_1 = (x_1, \cdots, x_6)$ satisfies

$$\sum_{\alpha=1}^{6} x_\alpha^2 = 0, \tag{5.14}$$

$X_1 = \pm\sqrt{-1}Y_1$, $X_2 = -Y_2$ and Z_2 depends linearly on X_2. Moreover, for $2 \leq l \leq 6$,

$$p_{l^*} = \sum_{\alpha=1,p=1}^{6,5} (S_{\alpha p}^l + \pm\sqrt{-1}T_{\alpha p}^l) x_\alpha z_p + \text{ other terms.} \tag{5.15}$$

We may assume the displayed sum is nontrivial for $l = 2$ since 2-nullity is impossible by item (1) of Lemma 4.2. (5.14) and (5.15) imply that $p_{\tilde{0}} = p_{\tilde{2}} = 0$ cuts down one more dimension in $\mathscr{S}_\lambda$ to carve out an irreducible variety $\mathcal{F}_2$ by Lemma .9 in Appendix I, so that the lower bound in (5.12), which is now 9, is achieved.

To cut down one more dimension to reach 8 on the right hand side of (5.12), observe that if $\mathcal{F}_j$, the irreducible variety of $\mathscr{S}_\lambda$ cut out by $p_{\tilde{0}} = p_{\tilde{j}} = 0, 3 \leq j \leq 6$, is distinct from $\mathcal{F}_2$, then one more dimension cut can be achieved by Lemma .9 in Appendix I, so that the codimension 2 estimate holds.

So now, we must rule out the case that all $\mathcal{F}_j, 3 \leq j \leq 6$, are identical with $\mathcal{F}_2$. Suppose they were all identical. It would then follow by a similar argument as in (5.7) through (5.10) in Lemma 5.1 that the displayed part of $p_{\tilde{4}}, p_{\tilde{5}}, p_{\tilde{6}}$ in (5.15) are all zero. We could then employ the same arguments immediately following Lemma 5.2 as for $j = 3$, with obvious modifications while invoking item (2) of Lemma 4.2, to reach a contradiction. Thus, generic $\lambda \in \mathcal{C}$ satisfies the codimension 2 estimate.

CASE IV. For $j = 1$, Lemma 4.1 allows us to apply Lemma 3.2 to obtain (5.12), whose right hand side is 10 obtained by setting $p_{\tilde{0}} = p_{\tilde{2}} = 0$ as usual.

Now, not all $p_{\tilde{j}}, j \geq 3$ are multiples of $p_{\tilde{2}}$ when restricted to $\mathscr{S}_\lambda$; for otherwise, we can argue exactly as in (5.8), (5.9) and (5.10) to obtain $p_{\tilde{6}} = 0$ when restricted to $\mathscr{S}_\lambda$ so that the basis element $\tilde{n}_6$ is 1-null, which is impossible by Lemma 4.1. So we may assume $p_{\tilde{2}}$ and $p_{\tilde{3}}$ are linearly independent when restricted to $\mathscr{S}_\lambda$. Then employing the same arguments one more time we can conclude that we may assume $p_{\tilde{2}}, p_{\tilde{3}}, p_{\tilde{4}}$ are linearly independent when restricted to $\mathscr{S}_\lambda$. Lemma .8 in Appendix I then enables us to further cut down 2 more dimensions from the right hand side of (5.12), so that the codimension 2 estimate holds.

CASE V. Lastly, for $j = 0$, no bases being 0-null lets us utilize (5.12) whose right hand side is 11. We may assume $p_{\tilde{2}}, p_{\tilde{3}}, p_{\tilde{4}}$ (understood to be restricted to $\mathscr{S}_\lambda$ in the following) are independent to be in accordance with item (2) of Corollary 2.1. For otherwise, a nontrivial linear combination of each of the triples $(\tilde{n}_2, \tilde{n}_3, \tilde{n}_4)$, $(\tilde{n}_2, \tilde{n}_3, \tilde{n}_5)$, and $(\tilde{n}_2, \tilde{n}_3, \tilde{n}_6)$ would result in three independent normal directions n_1', n_2', n_3' for which the B and C blocks of the corresponding shape operators $S_{n_1'}, S_{n_2'}, S_{n_3'}$ are zero to violate Corollary 2.1. Lemma .7 in Appendix I implies that $p_3 = p_4 = 0$ now cuts down two more dimensions from the right hand side of (5.12), so that the codimension 2 estimate goes through when $\dim(\mathcal{C}) \leq 3$ (i.e., $c_j \geq 2, j = 0$). We assume henceforth that $\dim(\mathcal{C}) = 4$. If $p_{\tilde{5}}$ and $p_{\tilde{6}}$ (understood to be restricted to $\mathscr{S}_\lambda$ in the following) are both dependent on $p_{\tilde{2}}, p_{\tilde{3}}, p_{\tilde{4}}$, then as before after a basis change we may assume $p_{\tilde{5}}$ and $p_{\tilde{6}}$ are zero. However, this implies that, as in the ending arguments in Lemma 5.1, through λ there is a unique $\mathcal{Q}_2$ in the irreducible component $\mathcal{C}$ of $\mathcal{L}_0$ where λ belongs. Since $\dim(\mathcal{C}) = 4$, we see as before that any two such quadrics have nonempty intersection in $\mathcal{C}$, a contradiction. Hence, we may assume that $p_{\tilde{2}}, \cdots, p_{\tilde{5}}$ are linearly independent. Lemma .7 in Appendix I implies that $p_{\tilde{3}} = p_{\tilde{4}} = p_{\tilde{5}} = 0$ now cuts down three more dimensions from the right hand side of (5.12). That is, the codimension 2 estimate goes through.

It follows that the codimension 2 estimate holds for $k = 6$ if the generic rank of $r_\lambda \geq 5$ for $\lambda \in \mathcal{Q}_5$; the isoparametric hypersurface is thus the one constructed by Ozeki and Takeuchi. This is impossible. So, we conclude the following.

Lemma 5.3. *Let* $(m_+, m_-) = (7, 8)$. *Suppose the isoparametric hypersurface is not the one constructed by Ozeki and Takeuchi. Away from points of Condition A on* M_+, *given an orthonormal pair* (n_0, n_1) *of normal vectors of* M_+, *let* S_{n_0} *and* S_{n_1} *be normalized as in* (2.2), (2.4) *and* (2.5). *Then the rank of the* B_1 (*and* C_1) *of* S_{n_1} *is* ≤ 4 *for any choice of* n_0.

Note that by Corollary 3.2, a generic normal basis is respectively 4-null, 3-null, or 2-null if the generic rank is 4, 3, or 2.

We will in fact establish that the generic rank is 4 in the next section in Proposition 6.1.

6. Mirror points

Let $x_0 \in M_+$ and let $n_0, n_a, a = 1, \cdots, m_+$, be a normal basis of M_+ at x_0. Set

$$x_0^{\#} := n_0, \quad n_0^{\#} := x_0.$$

Of fundamental importance is that $x_0^{\#}$ is also a point on M_+ with the normal space $\mathbb{R}n_0^{\#} \oplus E_0$, where E_0 is the 0-eigenspace of the shape operator S_{n_0} at x_0, whose basis vectors are denoted by $e_p, p = 1, \cdots, m_+$. The 0-eigenspace of the shape operator $S_{n_0^{\#}}$ at $x_0^{\#}$ is spanned by $n_a, a = 1, \cdots, m_+$. Moreover, S_{n_0} at x_0 and $S_{n_0^{\#}}$ at $x_0^{\#}$ share the same $(+1)-$ and (-1)-eigenspaces E_+ and E_-, whose basis vectors are denoted by e_α and $e_\mu, 1 \leq \alpha, \mu \leq m_-$, respectively.

Accordingly, recall that at $x_0 \in M_+$, subject to the eignespace decomposition of S_{n_0}, we let

$$S^a_{\alpha\mu} := \langle S(e_\alpha, e_\mu), n_a\rangle, \quad S^a_{\alpha p} := \langle S(e_\alpha, e_p), n_a\rangle, \quad S^a_{\mu p} := \langle S(e_\mu, e_p), n_a\rangle,$$

where S is the 2nd fundamental form at x_0. In a parallel fashion, at $x_0^{\#} \in M_+$, subject to the eigenspace decomposition of $S_{n_0^{\#}}$, we let

$$S^p_{\alpha\mu} := \langle S^{\#}(e_\alpha, e_\mu), e_p\rangle, \; S^p_{\alpha a} := \langle S^{\#}(e_\alpha, n_a), e_p\rangle, \; S^p_{\mu a} := \langle S^{\#}(e_\mu, n_a), e_p\rangle,$$

where $S^{\#}$ is the second fundamental form at $x_0^{\#}$, with the property [**5**, (1), (2), p. 472]

$$S^p_{\alpha a} = S^a_{\alpha p}, \quad S^p_{\mu a} = -S^a_{\mu p}.$$

Note that in general $S^p_{\alpha\mu} \neq S^a_{\alpha\mu}$ when a and p are equal in value in their respective ranges [**5**, (3), p. 473], which is the reason why the classification of isoparametric hypersurfaces can be entangling.

Referring to (2.2), where

$$A_a := \left(S^a_{\alpha\mu}\right), \quad B_a := \left(S^a_{\alpha p}\right), \quad C_a := \left(S^a_{\mu p}\right). \tag{6.1}$$

Let the counterpart matrices at $x_0^{\#}$ and their blocks be denoted by the same notation with an additional #. Then, for $p = 1, \cdots, m_+$,

$$A^{\#}_p := \left(S^p_{\alpha\mu}\right), \quad B^{\#}_p = \left(S^a_{\alpha p}\right), \quad C^{\#}_p = -\left(S^a_{\mu p}\right). \tag{6.2}$$

We call $x_0^{\#}$ the "mirror point" of x_0 on M_+.

Similarly, set

$$x_0^* := (x_0 + n_0)/\sqrt{2}, \quad n_0^* := (x_0 - n_0)/\sqrt{2}. \tag{6.3}$$

x_0^* is a point on M_- and n_0^* is normal to M_- at x_0^*. The normal space to M_- at x_0^* is $\mathbb{R}n_0^* \oplus E_+$. Furthermore, the $(+1)$-eigenspace E_+^* of the shape operator $S_{n_0^*}$ is spanned by $n_1, \cdots, n_{m_+}$, the (-1)-eigenspace E_-^* of $S_{n_0^*}$ is E_0, and the 0-eigenspace E_0^* of $S_{n_0^*}$ is E_-.

Referring to (2.2), let the counterpart matrices at x_0^* and their blocks be denoted by the same notation with an additional *. Then, for $\alpha = 1, \cdots, m_-$,

$$A_\alpha^* = -\sqrt{2}\left(S_{\alpha p}^a\right), \quad B_\alpha^* = -1/\sqrt{2}\left(S_{\alpha\mu}^a\right), \quad C_\alpha^* = -1/\sqrt{2}\left(S_{\alpha\mu}^p\right). \tag{6.4}$$

(Likewise, there are counterpart matrices when we replace α by μ at the points $(x_0^*)^\# \in M_-$.)

We call x_0^* the "mirror point" of x_0 on M_-. See [**4**, p. 144], [**5**, p. 474] for more details.

Proposition 6.1. *Notation as above, we may assume*

$$\begin{aligned} A_\alpha^* &= \begin{pmatrix} 0 & 0 \\ 0 & \cdot \end{pmatrix}, \quad B_\alpha^* = \begin{pmatrix} \cdot & 0 \\ 0 & \cdot \end{pmatrix}, \quad C_\alpha^* = \begin{pmatrix} \cdot & 0 \\ 0 & \cdot \end{pmatrix}, \quad 1 \leq \alpha \leq 4; \\ A_\alpha^* &= \begin{pmatrix} 0 & \cdot \\ \cdot & \cdot \end{pmatrix}, \quad B_\alpha^* = \begin{pmatrix} 0 & \cdot \\ \cdot & \cdot \end{pmatrix}, \quad C_\alpha^* = \begin{pmatrix} 0 & \cdot \\ \cdot & \cdot \end{pmatrix}, \quad 5 \leq \alpha \leq 8. \end{aligned} \tag{6.5}$$

In particular, Lemma 5.3 *can be improved to* 4*-nullity.*

Proof. By Lemma 5.3, a generic choice of x and $x^\#$ can only be r-null for $1 \leq r \leq 4$, so that the upper left $(8-r)$-by-$(7-r)$ block of $B_p^\#$ and $C_p^\#$ are zero for $1 \leq p \leq 7$. That is,

$$S_{\alpha p}^a = S_{\mu p}^a = 0, \quad 1 \leq \alpha, \mu \leq 8-r, 1 \leq a \leq 7-r, \forall p = 1, \cdots, 7. \tag{6.6}$$

In other words,

$$B_a = \begin{pmatrix} 0 & 0 \\ b_a & c_a \end{pmatrix}, \quad 1 \leq a \leq 7-r, \tag{6.7}$$

for some b_a, c_a, where the columns are indexed by p and the upper left block is of size $(8-r)$-by-$(7-r)$. (Likewise, C_a are of the same form.)

We normalize A_1 and B_1 as in (2.4) and (2.5). The proof of Corollary 2.1 implies that

$$A_a = \begin{pmatrix} z_a & 0 \\ 0 & \cdot \end{pmatrix}, \quad 2 \leq a \leq 7-r, \tag{6.8}$$

where the upper left block is of size $(8-r)$-by-$(8-r)$ with

$$z_a = -z_a^{tr}, \quad z_a z_b + z_b z_a = -2\delta_{ab} I, \quad 2 \leq a, b \leq 7-r. \tag{6.9}$$

That is, we have a Clifford C_{6-r}-module $\mathbb{R}^{8-r}$ for $1 \leq r \leq 4$; this is possible only when $r = 4$. In particular, generic points of M_+ are 4-null.

With $r = 4$ in place, note that, by (6.1) and (6.4), (6.8) is equivalent to

$$B^*_\alpha = \begin{pmatrix} h_\alpha & 0 \\ k_\alpha & \cdot \end{pmatrix}, \quad \alpha \leq 4; \quad B^*_\alpha = \begin{pmatrix} 0 & \cdot \\ \cdot & \cdot \end{pmatrix}, \quad 5 \leq \alpha \leq 8$$

for some h_α, k_α. Now the 4-nullity at x is

$$B_a = \begin{pmatrix} 0 & \cdot \\ \cdot & \cdot \end{pmatrix}, \quad \forall a = 1, \cdots, 7. \tag{6.10}$$

That is,

$$S^a_{\alpha p} = 0, \quad 1 \leq \alpha \leq 4, \quad 1 \leq p \leq 3, \quad \forall a = 1, \cdots, 7. \tag{6.11}$$

Putting (6.6) and (6.11) together, we obtain

$$A^*_\alpha = \begin{pmatrix} 0 & 0 \\ 0 & \cdot \end{pmatrix}, \quad 1 \leq \alpha \leq 4.$$

That the upper left corner of A^*_α is zero for $\alpha \geq 5$ is equivalent to that the lower left block of B_a in (6.7) is zero for $1 \leq a \leq 3$. To show the latter, item (1) of Corollary 2.1 implies that there is a matrix B_j, for some $j \geq 4$, of the form

$$B_j = \begin{pmatrix} 0 & d \\ b & c \end{pmatrix}, \quad d_{4\times 3} \neq 0. \tag{6.12}$$

Consider

$$E := uB_1 + vB_2 + wB_j = \begin{pmatrix} 0 & wd \\ v\beta + wb & u\sigma + v\gamma + wc \end{pmatrix}, \quad u^2+v^2+w^2 = 1,$$

where we set $\beta := b_2$ and $\gamma := c_2$ to suppress the index 2 for B_2 in (6.7). E is of rank at most 4, and is of rank 4 for u close to 1, so that the solution to the equation

$$\begin{pmatrix} 0 & wd \\ v\beta + wb & u\sigma + v\gamma + wc \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0,$$

is of dimension 3 for u close to 1. This amounts to

$$wdy = 0, \quad (v\beta + wb)x + (u\sigma + v\gamma + wc)y = 0.$$

Since $u\sigma + v\gamma + wc$ is invertible for u close to 1, we can solve y in terms the 3-dimensional x and insert it into $dy = 0$ (for small $w \neq 0$) to yield

$$d(u\sigma + v\gamma + wc)^{-1}(v\beta + wb) = 0,$$

whose Taylor expansion reads

$$d(I - (v'\sigma^{-1}\gamma + w'\sigma^{-1}c) + (v'\sigma^{-1}\gamma + w'\sigma^{-1}c)^2 - \cdots)\sigma^{-1}(v'\beta + w'b) = 0, \tag{6.13}$$

where $v' = v/u$ and $w' = w/u$, from which we can extract

$$d\sigma^{-1}\beta = 0. \tag{6.14}$$

That is, the column space of $\sigma^{-1}\beta$ is in the kernel of d. We thus conclude that

$$\text{the column space of } \sigma^{-1}\beta \subset \cap_{j=1}^{7} \operatorname{kernel}(d_j), \tag{6.15}$$

where

$$B_j := \begin{pmatrix} 0 & d_j \\ b_j & c_j \end{pmatrix}.$$

We claim that $\cap_{j=1}^{7}$ kernel(d_j) is of dimension at most 1. To this end, suppose the intersection is of dimension l. Reparametrizing, we may assume the first l columns of d_j are zero for all $j = 1, \cdots, 7$, which amounts to

$$S^a_{\alpha p} = 0, \quad 1 \leq \alpha \leq 4, \quad 4 \leq p \leq 3 + l, \quad \forall a = 1, \cdots, 7.$$

This is equivalent to

$$B_p^{\#} = \begin{pmatrix} 0 & 0 \\ \cdot & \cdot \end{pmatrix}, \quad p = 4, \cdots, 3 + l,$$

where the 0 rows are of size 4-by-7. On the other hand, (6.10) is equivalent to

$$B_p^{\#} = \begin{pmatrix} 0 & 0 \\ \cdot & \cdot \end{pmatrix}, \quad p = 1, 2, 3.$$

Therefore, normalizing $B_1^{\#}$ as in (2.4), we have that the top four rows of $B_j, 2 \leq j \leq 3 + l$, are zero. But then Corollary 2.1 implies that $l \leq 1$, because only Clifford C_2, when $l = 0$, and Clifford C_3, when $l = 1$, can act on $\mathbb{R}^4$. This proves the claim.

When $l = 0$, we have $\beta = 0$ by (6.15), i.e., the lower left block of B_a in (6.7) is zero for $1 \leq a \leq 3$. This is indeed true, as follows.

Sublemma 6.1. *$l = 0$ generically over M_+.*

Proof. Suppose that generically $l = 1$ over M_+. This is equivalent to saying, by considering generic x and $x^{\#}$, that there is an index $a \geq 4$, say, $a = 4$, and an index $p \geq 4$, say, $p = 4$, such that

$$S^{a=4}_{\alpha\, p} = S^{a}_{\alpha\, p=4} = 0, \quad 1 \leq \alpha \leq 4, \quad \forall a, p = 1, \cdots, 7.$$

That is, for each $\alpha \leq 4$, the first four columns and rows of the 7-by-7 matrix A^*_α in (6.5) are zero, i.e.,

$$A^*_\alpha = \begin{pmatrix} 0 & 0 \\ 0 & \delta_\alpha \end{pmatrix}, \quad 1 \leq \alpha \leq 4, \tag{6.16}$$

where δ_α is of size 3-by-3.

Note that in (6.9) we may assume that z_1, z_2 and z_3 are respectively the matrix representation of the quaternionic multiplication of the basis elements $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ on the left of $\mathbb{H}$; in doing so we do not assume $z_1 = I$ so that the representation will be notationally more consistent, and it will not affect the subsequent arguments. Accordingly ,we have

$$z_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; z_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \; z_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

according to which

$$h_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}/\sqrt{2}, \quad h_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}/\sqrt{2},$$
$$h_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}/\sqrt{2}, \quad h_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}/\sqrt{2}.$$

Moreover, we have (in B^*_α)

$$h_\alpha k_\alpha^{tr} = 0, \quad h_\alpha h_\alpha^{tr} = I/2, \quad \alpha \leq 4,$$

by (2.3a) when we set $i = j = \alpha$, where h_α is of size 3-by-4 and k_α is of size 4-by-4, from which we read off that the only possibly nonzero column of k_α is the αth one, i.e.,

$$k_\alpha = \left(\epsilon^\alpha_{jl}\delta_{l\alpha}\right), \quad 1 \leq \alpha, j, l \leq 4,$$

for some $\epsilon^\alpha_{jl} \in \mathbb{R}$. Now (2.3a) applied to $1 \leq \alpha \neq \beta \leq 4$ gives

$$h_\alpha k_\beta^{tr} + h_\beta k_\alpha^{tr} = 0,$$

which implies the four possibly nonzero columns are all identical, i.e,

$$\epsilon^1_{j1} = \epsilon^2_{j2} = \epsilon^3_{j3} = \epsilon^4_{j4}, \quad 1 \leq j \leq 4. \tag{6.17}$$

By performing a coordinate change on the a-indexes, $4 \leq a \leq 7$, indexing the rows of B^*_a, we may assume only the first components of these four columns are possibly nonzero, i.e.,

$$\epsilon^1_{j1} = \epsilon^2_{j2} = \epsilon^3_{j3} = \epsilon^4_{j4} = 0, \quad 2 \leq j \leq 4. \tag{6.18}$$

The same holds for $C^*_\alpha, \alpha \leq 4$, as well by changing the p-indexes, $4 \leq p \leq 8$. In fact, (2.3f) implies that we may further assume the nonzero entries of these columns for both B^*_α and C^*_α are identical.

Now, (2.3a) with $i = j = \alpha \leq 4$, we derive

$$B^*_\alpha (B^*_\alpha)^{tr} = \begin{pmatrix} I/2 & 0 \\ 0 & D_\alpha \end{pmatrix}, \quad D_\alpha = \begin{pmatrix} 1/2 & 0 \\ 0 & e_\alpha \end{pmatrix}, \tag{6.19}$$

where e_α is of size 3-by-3, in light of (6.16). Thus we may rearrange indexes (see [**1**, Lemma 49, p. 64]) to assume

$$A_\alpha^* = \begin{pmatrix} 0 & 0 \\ 0 & \delta_\alpha \end{pmatrix}, \quad B_1^* = C_1^* = \begin{pmatrix} 0 & I/\sqrt{2} & 0 \\ 0 & 0 & \sqrt{D} \end{pmatrix}, \quad \alpha \leq 4, \tag{6.20}$$

where $\sqrt{D}$ is diagonal of the form

$$\sqrt{D} = \mathrm{diag}(1/\sqrt{2}, 1/\sqrt{2}, b, b), \tag{6.21}$$

given the spectral data (σ, Δ) since δ_α is of size 3-by-3, where I is 3-by-3.

Suppose $\sqrt{D}$ is nonsingular. δ_1 is skew-symmetric as it is part of Δ. But then because nonsingularity of D is a generic condition, it follows that each linear combination of $\delta_\alpha, \alpha \leq 4$, is skew-symmetric of size 3-by-3 when suitably normalized, which implies that generic linear combinations of δ_α are of rank 2, from which we see, for a generic point $c := (c_1, \cdots, c_4) \in S^3$,

$$\delta_c := c_1\delta_1 + \cdots + c_4\delta_4,$$

that there is a unique c' on S^2 which is the eigen direction of δ_c with eigenvalue 0.

Without loss of generality, let us assume the map

$$F : S^3 \to S^2, \quad c \to c'$$

is surjective (more precisely, the domain and target spaces of F are projective spaces, though this does not create a problem); if F is not surjective the preimage will be of even larger dimension to our advantage. Then the closure $\mathcal{C}$ of the preimage $F^{-1}(c')$ is a 1-dimensional circle, because for $c \in F^{-1}(c')$, the plane perpendicular to c', which is an eigenspace of $\delta_c(\delta_c)^{tr}$, is fixed, from which we conclude that there is a unique point c_0 on $\mathcal{C}$ for which $\delta_{c_0} = 0$, because the spectral data stipulate that all δ_c for $c \in F^{-1}(c')$ be of the same form as δ_4 below. This means that we have an S^2 worth of δ_{c_0}, one for each c', which are identically zero, so that we may assume

$$\delta_1 = \delta_2 = \delta_3 = 0, \quad \delta_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}$$

for some $0 < \tau \leq 1/\sqrt{2}$. But then this implies that the first five columns and rows of $A_\alpha, \alpha \leq 4$, are zero, which contradicts $l \leq 1$.

On the other hand, suppose generic D_α is singular, then

$$D_\alpha \sim \mathrm{diag}(1/2, 1/2, 1/2, 0), \quad \text{or} \;\; \mathrm{diag}(1/2, 1/2, 0, 0).$$

If it is the former case, then δ_c has a 2-dimensional eigenspace with eigenvalue 0. Let us denote by c' the direction that is perpendicular to the 2-dimensional 0-eigenspace of δ_c; the spectral data stipulate that δ_c be of the form

$$\delta_c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & x \end{pmatrix}, \quad x^2 = 1.$$

We are done by the same reasoning as in the nonsingular case. If it is the latter, then e_c, whose components are given in the second matrix in (6.19), serves the role of δ_c in the former case, from which we conclude that there are $e_c = 0$, contradicting the given nonzero spectral data.

In conclusion, $l = 0$ generically. q.e.d.

As a consequence of the sublemma, the lower left block of $B_a = 0$ in (6.7) for $1 \leq a \leq 3$, or equivalently, the upper left corner of $A^*_\alpha = 0$ for $\alpha \geq 5$, for a generic choice of x and $x^{\#}$.

We will show in Lemma 6.1 below that the lower left blocks of B^*_α (and C^*_α), $\alpha \leq 4$, are zero through the next two sublemmas.

q.e.d.

Remark 6.1. *Intrinsically, in the preceding corollary, let $N^* \simeq \mathbb{H} \subset E_+$ be the kernel of B_1^{tr}, let $V_0^* \simeq \mathbb{H} \subset E_-^*$ be the kernel of C_1^{tr}, let $V_-^* \simeq Im(\mathbb{H}) \subset E_-^*$ be the kernel of B_1, and let $V_+^* \simeq Im(\mathbb{H}) \subset E_+^*$ be the kernel of $B_1^{\#}$. Then these four spaces parametrize the upper left blocks of the matrices in the corollary, where N^* is parametrized by $1 \leq \alpha \leq 4$, V_0^* by $1 \leq \mu \leq 4$, V_+^* by $1 \leq a \leq 3$, and V_-^* by $1 \leq p \leq 3$.*

Sublemma 6.2. *Notation as in the preceding remark, let*

$$V := V_+^* \oplus V_-^* \oplus V_0^* \subset E_+^* \oplus E_-^* \oplus E_0^* := E. \tag{6.22}$$

Let $p_j^*|_V$ and $q_j^*|_V, 0 \leq j \leq m_- = 8$ be the components of the second and third fundamental forms of M_- at x^* evaluated on V, where the indexes $1 \leq j \leq 4$ range through N^*, and as always $j = 0$ indexes the components corresponding to n_0^*. Then we have

$$\begin{aligned} p_j^*|_V &= 0, \quad j \geq 5, \\ q_j^*|_V &= 0, \quad 0 \leq j \leq 4. \end{aligned} \tag{6.23}$$

Proof. The first identity follows from the vanishing of the upper left blocks of the last three matrices in the statement of Corollary 6.1.

The second follows from the normal covariant derivative of the second fundamental form S^* at $x^* \in M_-$

$$\sum_k (S^*)^b_{ij;k}\, \omega^k = d(S^*)^b_{ij} - \sum_t (S^*)^b_{tj}\, \theta^t_i - \sum_t (S^*)^b_{it}\, \theta^t_j, \tag{6.24}$$

where $(S^*)^b_{ij;k}$ are the components of q^*_b, we assume the normal connection is zero at the point of calculation, and ω^j and θ^j_i are the coframe and connection forms.

We indicate one calculation for illustration. Let indexes $i, j \leq 3$ and $k \leq 4$ denote respectively those for E^*_+, E^*_- and E^*_0. Then for $1 \leq b \leq 4$, the right hand side of (6.24) is zero by the vanishing blocks of the first matrix in (6.5), knowing that $(S^*)^b_{uv} = 0$ whenever u and v index the same eigenspace and that θ^k_i and θ^k_j vanish on E^*_0 (see [**1**, (4.18), p. 14] for how to calculate θ^j_i in general).

On the other hand, the cubic polynomial

$$q^*_0|_V = \sum_{p\leq 4, i,j\leq 3} (S^*)^p_{ij}\, z_p\, x_i\, y_j = 0, \tag{6.25}$$

where p indexes the corresponding normal directions at $(x^*)^\#$, the mirror point of x^* on M_-, and $i, j \leq 3$ index the E^*_+ and E^*_-, respectively. The vanishing of the identity follows from that of the upper left block of the first matrix in (6.5) when we replace α by μ. q.e.d.

Sublemma 6.3. *Let* $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ *be the standard basis in* $\mathbb{H}$. *Write*

$$v = x \oplus y \oplus z$$

respecting the direct sum of V *in* (6.22), *and write*

$$p^* := p^*_1|_V\, \mathbf{1} + p^*_2|_V\, \mathbf{i} + p^*_3|_V\, \mathbf{j} + p^*_4|_V\, \mathbf{k}.$$

Then

$$p^*(v, v) = -\sqrt{2}(xz + y \circ z), \tag{6.26}$$

where $y \circ z = yz$ *or* zy *(quaternion multiplication).*

Proof. This follows from (6.9) and the corresponding identity for the matrix $A^\#_p, 1 \leq p \leq 3$. See [**4**, Remark 1, p. 140, and Proposition 1, p. 146] for more details. q.e.d.

Lemma 6.1. $q^*_j|_V = 0, \forall j$. *In particular, the lower left blocks of* B^*_α *and* $C^*_\alpha, \alpha \leq 4$, *in* (6.5) *are zero, as said in the end of the proof of Proposition* 6.1.

Proof. By the identity [**28**, (3-8), p. 530]

$$16(\sum_{a=0}^{8}(q^*_a)^2) = 16G(\sum_i u_i^2) - \langle \nabla G, \nabla G\rangle,$$

where $G := \sum_{a=0}^{8}(p^*_a)^2$ and u_i parametrize the tangential directions at x^*. (In [**28**, (2.1), p. 518], the notation $< G, G > := \langle \nabla G, \nabla G\rangle$

is used instead.) A straightforward calculation by the first identity in (6.23), (6.17), and (6.18) gives

$$
\begin{aligned}
16(\sum_{a=0}^{8}(q_a^*|_V)^2) = {} & 16G|_V(|x|^2+|y|^2+|z|^2) - \langle \nabla(G|_V), \nabla(G|_V)\rangle \\
& - 4c^2(\sum_{a=1}^{4} p_a^*|_V\, z_a)^2,
\end{aligned}
\tag{6.27}
$$

where x, y, z are given in the preceding corollary, $c = (S^*)^a_{5a}, 1 \le a \le 4$, and the factor 4 comes from the contribution of the $(5, a)$-entries, which are equal in value, of both B^*_α and $C^*_\alpha, \alpha \le 4$, in (6.5) (see also (6.17) and (6.18)).

In (6.26), if

$$p^*(v, v) = -\sqrt{2}(xz + zy), \tag{6.28}$$

then the sum of the first two terms on the right hand side of (6.27) vanishes, because it is exactly equal to the normed square of the third fundamental form of the **homogeneous** isoparametric hypersurface with multiplicity pair $(3, 4)$, which is zero. But then (6.27) implies that $c = 0$ and $q_a^*|_V = 0$ for all $0 \le a \le 8$.

Sublemma 6.4. *In* (6.26)*, it is impossible that*

$$p^*(v, v) = -\sqrt{2}(xz + yz). \tag{6.29}$$

Proof. Assume the contrary. Then the sum of the first two terms on the right hand side of (6.27) is

$$|xy - yx|^2|z|^2,$$

since it is the normed square of the third fundamental form of the **inhomogeneous** isoparametric hypersurface with multiplicity pair $(3, 4)$. Setting $x = y$ in (6.29), we obtain once more that $c = 0$, because $p^*(v, v) = -2\sqrt{2}xz$ makes the last term on the right hand side of (6.27) nonzero if $c \ne 0$, which is impossible.

In particular, the lower left blocks of B^*_α and $C^*_\alpha, \alpha \le 4$, in (6.5) are zero.

Now that

$$16(\sum_{a=0}^{8}(q_a^*|_V)^2) = |xy - yx|^2|z|^2$$

in the latter case, we see by the second identity of (6.23) that

$$16(\sum_{a=5}^{8}(q_a^*|_V)^2) = |xy - yx|^2|z|^2. \tag{6.30}$$

We will derive a contradiction. First, observe that (6.30) implies that $q_a^*|_V, a \geq 5$, are all multilinear in x, y, z and in fact after a coordinate change of z we may assume

$$q_5^*|_V \mathbf{1} + q_6^*|_V \mathbf{i} + q_7^*|_V \mathbf{j} + q_8^*|_V \mathbf{k} = (xy - yx)z. \tag{6.31}$$

This is because setting $x = y$ in (6.30), we see each $q_a^*|_V, a \geq 5$, is skew-symmetric in x and y and linear in z, so that $q_a^*|_V$ are of the form

$$q_a^*|_V =$$
$$(x_2y_3 - x_3y_2)\sum_b c_{1b}^a z_b + (x_3y_1 - x_1y_3)\sum_b c_{2b}^a z_b + (x_1y_2 - x_2y_1)\sum_b c_{3b}^a z_b,$$

for $1 \leq b \leq 4, 5 \leq a \leq 8$, where

$$xy - yx = (x_2y_3 - x_3y_2)\mathbf{i} + (x_3y_1 - x_1y_3)\mathbf{j} + (x_1y_2 - x_2y_1)\mathbf{k}.$$

The right hand side of (6.30) then asserts that the three 4-by-4 matrices $\left(c_{ib}^a\right), 1 \leq i \leq 3, 1 \leq a, b \leq 4$, form a Clifford system, and hence there follows (6.31). So now

$$\begin{aligned} &q_a^* = \langle (xy - yx)z, f_a \rangle + \\ &\textit{terms that involve at least one variable beyond those of } x, y, z, \end{aligned} \tag{6.32}$$

for $a \geq 5$, where

$$(f_5, f_6, f_7, f_8) := (\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}),$$

while for $a \geq 5$,

$$p_a^* \textit{ has no terms with only variables of } x, y, z,$$

by the first identity in (6.23). Meanwhile, by the block form of $B_\alpha^*, \alpha \leq 4$, in (6.5) we see

p_a^* *consists of terms with only variables of* x, z (or y, z) *and of terms with only variables beyond those of* x, y, z,

for $1 \leq a \leq 4$. Therefore, from the identity [**28**, (3-7), p. 529]

$$\sum_{a=0}^{8} p_a^* q_a^* = 0, \tag{6.33}$$

we deduce, when we set

$$(e_1, e_2, e_3, e_4) := (\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k})$$

and substitute (6.32), that

$$\sum_{a=5}^{8} \langle (e_b e_c - e_c e_b) e_p, f_a \rangle \, S_{b,c'}^a = 0, \tag{6.34}$$

where we set $x = e_b, y = e_c, z = e_p, 2 \leq b, c \leq 4, 1 \leq p \leq 4$, and $c' \geq 5$, and $\left(S_{bc'}^a\right)$ represents the upper right block of $A_a^*, a \geq 5$, in (6.5). Here, we also make use of the fact that for $1 \leq i \leq 4$, q_i^* has no terms involving both variables of x and y, while q_0^* has no terms involving both x and

z (or y and z), together with a third variable beyond x, y, z in either case, so that it is not a possibility to cancel the left hand side of (6.34) by the first five terms in (6.33); this follows from (6.24), (6.25) without the restriction to V, and the matrix types in (6.5). Consequently, we derive

$$S^a_{b,c'} = 0, \quad a, c' \geq 5, b \leq 4,$$

and likewise,

$$S^a_{b',c} = 0, \quad a, b' \geq 5, c \leq 4;$$

that is, the only possibly nonzero blocks of $A^*_\alpha, \alpha \geq 5$, in (6.5) are at the lower right corner.

$$A^*_\alpha = \begin{pmatrix} 0 & 0 \\ 0 & w_\alpha \end{pmatrix}, \quad \alpha \geq 5.$$

But then (6.24) establishes that

$$q^*_a|_V = 0, \quad a \geq 5.$$

This is a contradiction to (6.32). q.e.d.

Hence, we conclude by the sublemma that only (6.28) is valid, and thus $q^*_a|_V = 0$ for all $0 \leq a \leq 8$. q.e.d.

Corollary 6.1. *Let M be an isoparametric hypersurface with multiplicity pair $(m_+, m_-) = (7, 8)$ not constructed by Ozeki and Takeuchi. Given any point $p \in M$ with its unit normal n and any vector v at p tangent to a curvature surface (which is a sphere) of dimension* 7*, there is a* 16*-dimensional Euclidean space passing through p, n and v such that it cuts M in a homogeneous isoparametric hypersurface Z with multiplicity pair $(m_+, m_-) = (3, 4)$ in the* 15*-dimensional sphere.*

Proof. Notation as above, the 16-dimensional Euclidean space is just $\mathbb{R}x^* \oplus \mathbb{R}n^* \oplus V$, where x^* and n^* are given in (6.3) and V is given in (6.22), whose existence is generically established in Proposition 6.1, where p and n span the same plane as x^* and n^*_0, or as x and n_0, and v is the vector n_1 in the normal basis $n_0, n_1, \cdots, n_7$ at the focal point x with the normalization given in (2.4) and (2.5). The homogeneity of the resulting manifold Z follows from Lemma 6.1. Taking limit, the existence of the 16-dimensional Euclidean space is established everywhere.

q.e.d.

The preceding corollary points to that the isoparametric hypersurface should be one of the two constructed by Ferus, Karcher, and Münzner. We will prove in the next section that this is indeed the case.

7. The hypersurface is one constructed by Ferus, Karcher, and Münzner

When both x and $x^{\#}$ are generic in M_+ with the chosen 4-nullity bases as specified in Remark 6.1, it is more convenient to consider the conversion of (6.5) from x^* to x to obtain

(7.1)
$$A_a = \begin{pmatrix} z_a & 0 \\ 0 & w_a \end{pmatrix}, \quad B_a = \begin{pmatrix} 0 & 0 \\ 0 & c_a \end{pmatrix}, \quad C_a = \begin{pmatrix} 0 & 0 \\ 0 & f_a \end{pmatrix}, \quad 1 \le a \le 3,$$
$$A_a = \begin{pmatrix} 0 & \beta_a \\ \gamma_a & \delta_a \end{pmatrix}, \quad B_a = \begin{pmatrix} 0 & d_a \\ b_a & c_a \end{pmatrix}, \quad C_a = \begin{pmatrix} 0 & g_a \\ b_a & f_a \end{pmatrix}, \quad 4 \le a \le 7.$$

Observe that the matrices $\begin{pmatrix} \sqrt{2}c_a & w_a \end{pmatrix}$, $1 \le a \le 3$, form a Clifford multiplication of type $[3, 4, 8]$.

(7.2) $$F : \mathbb{R}^3 \times \mathbb{R}^4 \to \mathbb{R}^8, \quad F(u_a, v_\alpha) = \text{ the } \alpha\text{th row of } \begin{pmatrix} \sqrt{2}c_a & w_a \end{pmatrix}.$$

This is the starting point of our remaining task to pinpoint the characteristic features of the undetermined blocks of the matrices in (7.1). In [**8**], we have classified the orthogonal multiplications of type $[3, 4, 8]$, which we will apply to understand (7.1).

Lemma 7.1. *Given four 4-by-3 matrices $b_i, 4 \le i \le 7$, consider the linear combinations*
$$b(x) := x_1 b_4 + \cdots + x_4 b_7.$$
Suppose the first column of $b(x)$ is
$$x = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^{tr}$$
(*more generally, suppose the four components of the first column are linearly independent linear polynomials*), *and suppose generic $b(x)$ is of rank $= 2$. Then we may assume, e.g., the third columns of $b_i, 4 \le i \le 7$, are zero after a simultaneous column operation, i.e., the three column vectors of b_i are subject to the same linear constraint for all $4 \le i \le 7$.*

Proof. This follows from the fact that the Koszul complex
$$0 \longrightarrow R \xrightarrow{x\wedge} \Lambda^1 R^4 \xrightarrow{x\wedge} \Lambda^2 R^4 \xrightarrow{x\wedge} \Lambda^3 R^4 \xrightarrow{x\wedge} \Lambda^4 R^4 \to 0,$$
where $R := \mathbb{R}[x_1, x_2, x_3, x_4]$ is the polynomial ring in four variables and $x\wedge$ means taking the wedge product against x, is a free resolution. The assumption that $b(x)$ is generically of rank 2 means that the wedge product of second column v_2 and third column v_3 of $b(x)$ lives in the kernel of
$$\longrightarrow \Lambda^2 R^4 \xrightarrow{x\wedge} \Lambda^3 R^4, \quad v_2 \wedge v_3 \mapsto x \wedge (v_2 \wedge v_3) = 0,$$
so that either $v_2 \wedge v_3 = 0$, in which case they differ by a constant multiple, or $v_2 \wedge v_3 = x \wedge w$ for some $w \in R^4$, so that we may assume

the first two columns of $b(x)$ are both x up to a constant multiple. q.e.d.

Remark 7.1. *When the generic rank of $b(x)$ is 1, it is clear that two column vectors of $b(x)$ are constant multiples of the remaining one because all entries are linear.*

Lemma 7.2. *Assume the isoparametric hypersurface is not of the type constructed by Ozeki and Takeuchi. Away from points of Condition A in M_+, let (n_0, n_1) be 4-null with the decomposition in (6.5) (expressed over M_- with the conversion to the corresponding data over M_+ by (6.1), (6.2), (6.4)). Then for $4 \le a \le 7$ over M_+, the generic linear combination of the 4-by-3 matrices b_a in*

$$B_a = \begin{pmatrix} 0 & d_a \\ b_a & c_a \end{pmatrix}$$

is of rank ≤ 2, so that by Lemma 7.1 we may assume $b_a, 4 \le a \le 7$, share a common zero column. As a consequence, the spectral data (σ, Δ) is such that $\sigma = s\,Id$ for some $s > 0$.

Proof. At generic x and $x^{\#}$ in M_+ with 4-nullity, b_a cannot be all zero for $4 \le a \le 7$ at x. Otherwise, translated to the data at $x^{\#}$ by (6.1) and (6.2) (see the following **NOTE**), the matrices $B_p^{\#}, 1 \le p \le 3$, which are of the form

$$B_p^{\#} = \begin{pmatrix} 0 & 0 \\ 0 & c_p^{\#} \end{pmatrix}, \tag{7.3}$$

would be such that $c_p^{\#} = 0, 1 \le p \le 3$, which contradicts the 4-nullity of $B_1^{\#}$.

NOTE:: *When viewed at $x^{\#}$ the first columns of $b_4, \cdots, b_7$ are, respectively, the first, second, third, and fourth columns of $c_1^{\#}$, i.e.,*

$$c_1^{\#} = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_1 & 0 & 0 \\ 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & 0 & \sigma_2 \end{pmatrix},$$

in (7.3). Similarly, the second (vs. third) columns of $b_4, \cdots, b_7$ are the respective columns of $c_2^{\#}$ (vs. $c_3^{\#}$).

Returning to the proof, suppose, e.g., b_4 is of rank 3. Since

$$d_4 \sigma^{-1} b_4 = 0, \tag{7.4}$$

which holds by an analysis similar to the one following (6.13), d_4 is perpendicular to the 3-dimensional column space of $\sigma^{-1} b_4$. Hence by row operations without changing the spectral data in the normalized B_1, we may assume the only nonzero row of d_4 is the first one.

We claim that $c_4 = f_4$. To prove the claim, observe that we have

$$\sigma(c_4 - f_4) = -(c_4 - f_4)^{tr}\sigma, \quad b_4^{tr}(c_4 - f_4) = 0,$$

which are (3.19d) and the first equation of (4.6), which together with the fact that b_4 is of rank 3 force $c_4 - f_4 = 0$. It follows that

$$d_4^{tr} d_4 = g_4^{tr} g_4$$

by the second equation of (4.6), so that g_4 is of the same rank as d_4, which is ≤ 1. Now the formula $A_4 A_4^{tr} + 2B_4 B_4^{tr} = I$ gives

$$\beta_4 \beta_4^{tr} + 2d_4 d_4^{tr} = Id,$$

where as usual

$$A_4 = \begin{pmatrix} 0 & \beta_4 \\ \gamma_4 & \delta_4 \end{pmatrix},$$

so that $\beta_4 \beta_4^{tr} = I - 2d_4 d_4^{tr}$ is diagonal of rank at least 3 since the only nonzero row of d_4 is the first one. But then the identity

$$g_4 \sigma^{-1} = d_4 \sigma^{-1} \Delta - \beta_4,$$

which is (3.19a), gives that g_4 is of rank at least 3. This is a contradiction.

It follows that the generic rank of linear combination

$$b(x) := x_1 b_4 + \cdots + x_4 b_7$$

is ≤ 2, so that by Lemma 7.1 we may assume a fixed column of $b_4, \cdots, b_7$ is identically zero. Note that the condition in Lemma 7.1 that the four components of the first column of $b(x)$ are linearly independent linear polynomials is satisfied by a look at $c_1^{\#}$ in the above **NOTE**. Therefore, when viewed at $x^{\#}$, we conclude by Lemma 7.1 that one of the $c_p^{\#}$, is identically zero; we may assume $c_3^{\#} = 0$.

It follows from [**8**, Proposition 1, Proposition 2, Corollary 2, Remark 3] that

$$\sigma_1 = \sigma_2 := \epsilon, \quad \text{so,} \quad c_1^{\#} = \epsilon\, Id. \tag{7.5}$$

Indeed, in the terminology of [**8**], our orthogonal multiplication lives in the "grand moduli" (see the following remark) of the moduli space of orthogonal multiplications of type $[3, 4, 8]$, in which $\sigma_1 = \sigma_2$; it is a consequence, following the notation of [**8**], that $\mu = -\nu$ for which $\sigma_1^2 = 1 - \mu^2$ and $\sigma_2^2 = 1 - \nu^2$, where μ and ν are certain inner products associated with the orthogonal multiplication; see the following remark for the precise definition of μ and ν.

Swapping $x^{\#}$ and x by a symmetric argument, we conclude that $c_1 = s\, Id$ for some $s > 0$. q.e.d.

Remark 7.2. *We spell out in some details the essentials in* [**8**] *for easier reading. Let* $F : \mathbb{R}^3 \times \mathbb{R}^4 \to \mathbb{R}^8$ *be an orthogonal multiplication, where* $\mathbb{R}^3$ *is spanned by an orthonormal basis* $u_i, 1 \le i \le 3$, *and* $\mathbb{R}^4$ *is spanned by an orthonormal basis* $v_j, 1 \le j \le 4$. *Set*

$$u_i \circ v_j := F(u_i, v_j), \quad F_{ij,kl} := \langle u_i \circ v_j, u_k \circ v_l \rangle,$$

where $\langle \cdot, \cdot \rangle$ *denotes the inner product of* $\mathbb{R}^8$. *Then* F *being orthogonal implies that*

$$\begin{aligned} &F_{ij,kl} = F_{kl,ij}, \quad \forall i,j,k,l,\\ &F_{ij,il} = F_{li,ji} = 0, \quad j \neq l,\\ &F_{ij,kl} = -F_{il,kj}, \quad i \neq k,\\ &F_{ij,kl} = -F_{kj,il}, \quad j \neq l, \end{aligned} \tag{7.6}$$

from which there follows that

$$[ij,kl] := F_{ik,jl}$$

is skew-symmetric in (i,j) *and* (k,l) *(see* [**8**, (1), (2)]). *In other words, we can define the matrix*

$$C : \wedge^2(\mathbb{R}^4) \to \wedge^2(\mathbb{R}^3), \quad C := \big([ij,kl]\big).$$

Since $\wedge^2(\mathbb{R}^4) \simeq \wedge^2(\mathbb{R}^3) \oplus \wedge^2(\mathbb{R}^3)$, *we may assume, after a coordinate change of* $\mathbb{R}^4$, *that the* 3*-by-*6 *matrix* C *is of the form* $C = \begin{pmatrix} A & B \end{pmatrix}$, *where* A *and* B *are of size* 3*-by-*3 *and* A *is diagonal and* B *is upper triangular. This sets up certain constraints on* $F_{ij,kl}$ *(see* [**8**, (12),(13)]),

$$\begin{aligned} &F_{31,12} = F_{33,14} = 0, \quad F_{31,22} = F_{33,24} = 0, \quad F_{31,23} = F_{34,22} = 0,\\ &F_{11,23} + F_{14,22} = F_{11,24} + F_{12,23} = F_{31,14} + F_{32,13} = 0. \end{aligned} \tag{7.7}$$

With the constraints we can introduce an appropriate orthonormal basis $\eta_1, \cdots, \eta_8$ *(denoted by* $u_1, \cdots, u_8$ *in* [**8**]) *relative to which, if we set*

$$F_a := \text{the matrix whose } b\text{th row is } u_a \circ v_b,$$

we have

$$F_a := \begin{pmatrix} c_a & w_a \end{pmatrix}, \quad 1 \le a \le 3,$$

where c_a *and* w_a *are of size* 4*-by-*4, *and, moreover,*

$$c_3 = 0, \quad w_3 = Id,$$

whereas

$$c_1 := \begin{pmatrix} \sigma_1 & 0 & 0 & 0\\ 0 & \sigma_2 & 0 & 0\\ 0 & 0 & \sigma_2 & 0\\ 0 & 0 & 0 & \sigma_1 \end{pmatrix}, \quad w_1 = \begin{pmatrix} 0 & 0 & 0 & -\mu\\ 0 & 0 & -\nu & 0\\ 0 & \nu & 0 & 0\\ \mu & 0 & 0 & 0 \end{pmatrix},$$

where

$$\mu = F_{24,31}, \quad \nu = F_{23,32}, \quad \sigma_1 = \sqrt{1-\mu^2}, \quad \sigma_2 = \sqrt{1-\nu^2},$$

and

$c_2 =$

$$\begin{pmatrix} -\beta\mu/\sigma_1 & (F_{11,22}-\alpha\nu)/\sigma_2 & F_{11,23}/\sigma_2 & F_{11,24}/\sigma_1 \\ (F_{12,21}-\mu\gamma)/\sigma_1 & \beta\nu/\sigma_2 & F_{12,23}/\sigma_2 & F_{12,24}/\sigma_1 \\ F_{13,21}/\sigma_1 & F_{13,22}/\sigma_2 & \beta\nu/\sigma_2 & (F_{13,24}-\alpha\mu)/\sigma_1 \\ F_{14,21}/\sigma_1 & F_{14,22}/\sigma_2 & (F_{14,23}-\gamma\nu)/\sigma_2 & -\beta\mu/\sigma_1 \end{pmatrix},$$

while

$$w_2 = \begin{pmatrix} 0 & 0 & -\alpha & -\beta \\ 0 & 0 & \beta & -\gamma \\ \alpha & -\beta & 0 & 0 \\ \beta & \gamma & 0 & 0 \end{pmatrix}, \quad \alpha := F_{31,13}, \quad \beta := F_{31,14}, \quad \gamma := F_{32,14}.$$

(*see* [**8**, (15),(16)]).

The moduli space of orthogonal multiplications of type $[3,4,8]$, *under the domain and range equivalence, is thus parametrized by the* $F_{ij,kl}$ *in the above data of* $F_a, 1 \le a \le 3$.

The **grand moduli** *of the moduli space of orthogonal multiplications of type* $[3,4,8]$ *is where* $\alpha = \gamma$ *and* $\mu = -\nu$, *in which case, on the one hand,*

$$\sigma_1 = \sigma_2 := s, \quad so, \quad c_1 = s\, Id, \tag{7.8}$$

which is exactly (7.5) *at* $x^{\#}$, *and, on the other hand,* (7.6) *and* (7.7) *applied to* c_2 *implies that*

$$c_2 = -\frac{\beta\mu}{s}\, Id + M,$$

in the case $s > 0$, *where* M *is skew-symmetric* (*see* [**8**, Remark 1]). *The Hurwitz condition*

$$c_2 c_2^{tr} + w_2 w_2^{tr} = Id$$

is reduced to

$$MM^{tr} = \theta^2\, Id, \quad \theta = \sqrt{1 - \frac{\beta^2\mu^2}{s^2} - \alpha^2}, \quad s = \sqrt{1-\mu^2}.$$

Therefore, with a coordinate change we may assume

$$c_2 = a\, Id + b \begin{pmatrix} I & 0 \\ 0 & \pm I \end{pmatrix}, \quad I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{7.9}$$

for some a *and* b.

In the isoparametric situation, the range equivalence of an orthogonal multiplication is under the more rigid $SO(4) \oplus SO(4)$ *equivalence, because* (7.1) *stipulates how the orthogonal multiplication is defined in* (7.2), *in which case the conclusions in* [**8**, Proposition 1, Proposition 2, Corollary 2, Remark 3] *ensure that* $b \neq 0$ *in* (7.9).

Remark 7.3. *We summarize before we proceed further. When both x and $x^{\#}$ are generic in M_+ with the chosen 4-nullity bases as specified in Remark* 6.1*, we have* (7.1) *where, interchanging x and $x^{\#}$ by symmetry, we may assume, by* (7.8) *and* (7.9)*, that*

$$c_1^{\#} = \epsilon\, I, \quad c_3^{\#} = 0$$

for some $\epsilon > 0$.

Moreover, the third columns of the four 4-by-3 matrices $b_4, \cdots, b_7$ at x are zero in accordance with $c_3^{\#} = 0$. By (7.9) *applied at $x^{\#}$, the matrix $c_2^{\#}$ is of the form*

$$c_2^{\#} = a\, Id + b \begin{pmatrix} I & 0 \\ 0 & \pm I \end{pmatrix}, \quad I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \ b \neq 0, \tag{7.10}$$

for some a and b; with $c_1^{\#} = \epsilon\, Id$ and $c_3^{\#} = 0$, the three matrices can be converted, by **NOTE** *in Lemma 7.2, to the data*

(7.11)

$$b_4 = \begin{pmatrix} \epsilon & a & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ b_5 = \begin{pmatrix} 0 & -b & 0 \\ \epsilon & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ b_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \epsilon & a & 0 \\ 0 & \pm b & 0 \end{pmatrix}, \ b_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mp b & 0 \\ \epsilon & a & 0 \end{pmatrix}$$

at x, whose linear combinations are of generic rank 2*.*

In particular, a glance at $B_a, 1 \leq a \leq 7$, in (7.1) *shows that their third columns are all zero, or equivalently, that there is a common kernel vector for all the shape operators S_n for all n.*

Lemma 7.3. *Let $(m_+, m_-) = (7, 8)$. Assume the isoparametric hypersurface is not the one constructed by Ozeki and Takeuchi. Then at each point of M_+ the intersections of the kernels of all the shape operators is nontrivial, which is generically of dimension* 1*.*

Proof. Lemma 7.2 and Remark 7.3 establish the existence of such a common eigenvector for generic points of M_+, and so the existence is true everywhere by taking limit. Generically the dimension of this common eigenspace must be 1-dimensional because generic linear combinations of $b_4, \cdots, b_7$ is of rank 2 as said in Remark 7.3. q.e.d.

Remark 7.4. *The preceding lemma gives us a clear geometric picture. Namely, when the isoparametric hypersurface with multiplicities $(m_+, m_-) = (7, 8)$ is not the one constructed by Ozeki and Takeuchi, consider the quadric $\mathcal{Q}_6$ of oriented 2-planes in the normal space at a generic point $x \in M_+$. We know a generic element (n_0, n_1) in $\mathcal{Q}_6$ is 4-null, or equivalently, the intersection V of the kernels of S_{n_0} and S_{n_1} is 3-dimensional. By the preceding lemma, there is a nonzero unit vector $v \in V$ common to all kernels of the shape operators at x. We choose an orthonormal basis $e_1, e_2, e_3 = v$ spanning V. When viewed at the mirror point $x^{\#} = n_0 \in M_+$, e_1, e_2, e_3 are converted to three normal* *basis vectors of which the three matrices $c_1^\#, c_2^\#, c_3^\#$ given in* (7.1) *are of the form $c_1^\# = \epsilon\, Id, c_3^\# = 0$, and $c_2^\#$ is given in* (7.10).

By a symmetric reasoning, all this holds true as well at x when both x and $x^\#$ are generic.

Lemma 7.4. *A generic linear combination*

$$d(x) := x_1 d_4 + \cdots + x_4 d_7$$

of $d_4, \cdots, d_7$ is of rank ≤ 2. In particular, we may assume the last two rows of $d(x)$ are zero.

Proof. $b(x)$ is of generic rank 2 by the preceding lemma, which is explicitly given in (7.11). On the other hand, similar to (7.4), we have

$$d(x)b(x) = 0 \tag{7.12}$$

(and similarly $g(x)b(x) = 0$), knowing now $\sigma = s\, Id$, so that each row $r_i(x), 1 \leq i \leq 4$, of $d(x)$ annihilates $b(x)$. Hence, it must be that

$$r_i(x) = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} M_i,$$

where M_i is a skew-symmetric constant matrix, because the first column of $b(x)$ is $\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}$, which is a regular sequence [**3**, (5), p. 7], [**6**, p. 93]. On the other hand, the same sort of relation must hold true for the second column of $b(x)$ as well. That is,

$$r_i(x) = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} \Gamma(\Gamma^{-1} M_i),$$

where $\Gamma^{-1} M_i$ is skew-symmetric,

$$\Gamma := \begin{pmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & a & \pm b \\ 0 & 0 & \mp b & a \end{pmatrix},$$

and $\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} \Gamma$ is the second column of $b(x)$ transposed in light of (7.11). It follows that

$$M_i = \begin{pmatrix} 0 & U \\ -U^{tr} & 0 \end{pmatrix}, \quad U := \begin{pmatrix} u & v \\ -v & u \end{pmatrix}.$$

Therefore, all four rows of $d(x)$ are linearly spanned by the two vectors

$$\begin{aligned} &\begin{pmatrix} -x_3 & -x_4 & x_1 & x_2 \end{pmatrix}, \\ &\begin{pmatrix} -x_4 & x_3 & -x_2 & x_1 \end{pmatrix}. \end{aligned} \tag{7.13}$$

. q.e.d.

Lemma 7.5. *With the condition that the last two rows of $d(x)$ are zero, we may assume the first two rows of $g(x)$ are zero.*

Proof. We know $\sigma = sI$ by Lemma 7.2 and

$$\Delta = \begin{pmatrix} \tau J & 0 \\ 0 & \tau J \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau = \sqrt{1-2s^2}$$

for some s. By (7.1) and the fact that

$$A(x)A(x)^{tr} + 2B(x)B(x)^{tr} = I,$$

where $A(x) = x_1 A_4 + \cdots + x_4 A_7$ and likewise for $B(x)$, it follows by comparing the upper left block of the involved matrices that we obtain

$$\beta(x)\beta^{tr}(x) + 2d(x)d(x)^{tr} = I. \tag{7.14}$$

We employ

$$\beta(x) = s^{-1}(d(x)\Delta - g(x)), \tag{7.15}$$

which is (3.19a), to derive

$$\begin{aligned} &s^2\beta(x)\beta(x)^{tr} \\ &= (d(x)\Delta - g(x))(d(x)\Delta - g(x))^{tr} \\ &= \tau^2 d(x)d(x)^{tr} + g(x)g(x)^{tr} - (d(x)\Delta g(x)^{tr} - g(x)\Delta d(x)^{tr}), \end{aligned}$$

so that with $\tau^2 = 1 - 2s^2$ and (7.14) we obtain

$$s^2 I = d(x)d(x)^{tr} + g(x)g(x)^{tr} - (d(x)\Delta g(x)^{tr} - g(x)\Delta d(x)^{tr}),$$

where the lower right 2-by-2 blocks of all the matrices on the right, except for $g(x)g(x)^{tr}$, are zero because the last two rows of $d(x)$ are zero. Therefore, the lower right 2-by-2 block of $g(x)g(x)^{tr}$ is $s^2 I$, which means that the last two rows of $g(x)$ are linearly independent. We can accordingly do row reductions to annihilate the first two rows of $g(x)$ by the last two while performing the same row reduction on $d(x)$ to not to change the spectral data, where in fact $d(x)$ is not affected by the row reduction since its last two rows are zero.

q.e.d.

Lemma 7.6. *The spectra data are* $(\sigma, \Delta) = (1/\sqrt{2}I, 0)$.

Proof. Employing that $d(x)$ and $g(x)$ are of the form

$$d(x) = \begin{pmatrix} d_1(x) & d_2(x) \\ 0 & 0 \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0 & 0 \\ g_1(x) & g_2(x) \end{pmatrix},$$

by the preceding lemma, we employ (7.14) and (7.15) to arrive at

$$\begin{aligned} &d_1(x)d_1(x)^{tr} + d_2(x)d_2(x)^{tr} = g_1(x)g_1(x)^{tr} + g_2(x)g_2(x)^{tr} = s^2 I, \\ &\tau(d_1(x)Jg_1(x)^{tr} + d_2(x)Jg_2(x)^{tr}) = 0, \quad x_1^2 + \cdots x_4^2 = 1. \end{aligned}$$

However, since $d_1(x)$ are in terms of x_3, x_4 and $d_2(x)$ are in terms of x_1, x_2, and likewise for $g_1(x)$ and $g_2(x)$, there must hold, by homogenizing,

$$\begin{aligned}&d_1(x)d_1(x)^{tr} = s^2(x_3^2 + x_4^2), \quad d_2(x)d_2(x)^{tr} = s^2(x_1^2 + x_2^2).\\&\tau d_1(x)Jg_1(x)^{tr} = 0 = \tau d_2(x)Jg_2(x)^{tr}.\end{aligned} \tag{7.16}$$

That is,

$$d_1 = sU\begin{pmatrix} -x_3 & -x_4 \\ x_4 & -x_3 \end{pmatrix}, \quad d_2 = sU\begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix} \tag{7.17}$$

for some 2-by-2 orthogonal matrix U; by the same token,

$$g_1 = sW\begin{pmatrix} -x_3 & -x_4 \\ x_4 & -x_3 \end{pmatrix}, \quad g_2 = sW\begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix} \tag{7.18}$$

with W orthogonal, which we substitute into the third equality of (7.16) to derive

$$0 = \tau U\begin{pmatrix} 0 & x_3^2 + x_4^2 \\ -(x_3^2 + x_4^2) & 0 \end{pmatrix} W^{tr}.$$

This is possible only when $\tau = 0$, i.e., when the spectral data $(\sigma, \Delta) = (I/\sqrt{2}, 0)$. q.e.d.

Corollary 7.1. *Notation as in* (7.1), *we have* $c_a = f_a, 1 \leq a \leq 7$, *and hence* $\delta_a, 1 \leq a \leq 7$, *are skew-symmetric.*

Proof. Let us first handle the case when $4 \leq a \leq 7$. We know $c_a - f_a$ is skew-symmetric by (3.19d) because the spectral data are $(\sigma, \Delta) = (I/\sqrt{2}, 0)$ now. Moreover,

$$(c_a - f_a)^{tr} b_a = 0$$

by (4.6). Hence linear combinations of $c_a - f_a, 4 \leq a \leq 7$, i.e.,

$$h(x) := x_1(c_4 - f_4) + \cdots + x_4(c_7 - f_7),$$

satisfies

$$h(x)b(x) = 0$$

and so the first row of $h(x)$ is a linear combination of the vectors in (7.13). However, since $h(x)$ is skew-symmetric, the first component of the first row of $h(x)$ is zero. Consequently, the entire first row of $h(x)$ is, and similarly, all rows of $h(x)$ are zero. That is, $c_a = f_a$ for all $4 \leq a \leq 7$.

For $1 \leq a \leq 3$, the first columns of $b_4, \cdots, b_7$ at x are placed in order to form the first matrix $c_1^{\#}$ and $f_1^{\#}$ at $x^{\#}$, the second columns to form $c_2^{\#}$ and $f_2^{\#}$, the third to form $c_3^{\#}$ and $f_3^{\#}$, and vice versa. It follws that $c_a = f_a, 1 \leq a \leq 3$, because they are both generated by aligning the columns of $b_4^{\#}, \cdots, b_7^{\#}$.

That δ_a is skew-symmetric follows from (3.19f) and $\Delta = 0$. Lastly,

$$d_a^{tr} d_a = g_a^{tr} g_a$$

follows from the second identity in (4.6). q.e.d.

We are in a position to prove the classification theorem.

Theorem 7.1. *Let* $(m_+, m_-) = (7, 8)$. *Assume the isoparametric hypersurface is not the one constructed by Ozeki and Takeuchi. Then the hypersurface is one of the two constructed by Ferus, Karcher, and Münzner.*

Proof. Referring to (6.5), we will show there is a **Clifford frame** [**1**, (8.1)-(8.4), p. 28] on the unit normal bundle of M_-.

Recall the tangent bundle $\mathcal{T}$ of the unit bundle $\mathcal{U}N$ of M_- naturally splits into the vertical part $\mathcal{V}$ and and the horizontal part $\mathcal{H}$, and $\mathcal{H}$ further splits into three subspaces which, at $(x^*, n^*) \in \mathcal{U}N$ sitting over $x^* \in M_-$, are the horizontal lift of the three eigenspaces of the shape operator S_{n^*} at x^* with eigenvalues $0, 1, -1$, respectively, i.e.,

$$\mathcal{T} = \mathcal{V} \oplus \mathcal{E}_0^* \oplus \mathcal{E}_+^* \oplus \mathcal{E}_-^*,$$

where the basis elements of $\mathcal{V}, \mathcal{E}_0^*, \mathcal{E}_+^*, \mathcal{E}_-^*$ are indexed by subscripts α, μ, a, p, where $1 \leq \alpha, \mu \leq 8, 1 \leq a, p \leq 7$, so that a typical one is denoted, respectively, by $e_\alpha, e_\mu, e_a, e_p$ in the corresponding range with dual frame $\omega^\alpha, \omega^\mu, \omega^a, \omega^p$ and connection forms θ^i_j with i, j ranging over all possible indexes; for a specific index in a range, we will denote it by, e.g., $e_{\alpha=5}, \theta^{a=6}_{\mu=5}$, etc. Write

$$\theta^i_j = \sum_k F^i_{jk} \omega^k. \tag{7.19}$$

We know [**1**, (2.9), p. 9] $F^i_{jk} = 0$ whenever exactly two indexes fall in the same α, μ, a, or p range.

A Clifford frame is one on $\mathcal{T}$ that satisfies

$$\begin{aligned}
&A^*_\alpha = A^*_\mu, \\
&(a, \mu) \text{ entry of } \; B^*_\alpha = -(a, \alpha) \text{ entry of } B^*_\mu, \\
&(p, \mu) \text{ entry of } \; C^*_\alpha = -(p, \alpha) \text{ entry of } C^*_\mu, \\
&\theta^i_j - \theta^{i'}_{j'} = \sum_k L^i_{jk}(\omega^k + \omega^{k'})
\end{aligned} \tag{7.20}$$

for some smooth functions L^i_{jk}, where i, j, k are in the α index range and i', j', k' are in the μ index range with the same respective index values (i.e., i indicates $\alpha = i$ and i' indicates $\mu = i$, etc.)

It was shown in [**1**] that a Clifford frame characterizes an isoparametric hypersurface constructed by Ozeki-Takeuchi and Ferus-Karcher-Münzner. Moreover, it is shown in [**2**] that a Clifford frame is the same as a distribution $\mathcal{D}$ over $\mathcal{T}$ given by

$$\mathcal{D} = \mathcal{F} \oplus \mathcal{E}_+^* \oplus \mathcal{E}_-^*,$$

where $\mathcal{F} \subset \mathcal{V} \oplus \mathcal{E}_0^*$ is the graph of an orthogonal bundle map

$$Q : \mathcal{E}_0^* \to \mathcal{V},$$

where we define

$$e_{\alpha=j} := -Q(e_{\mu=j}), \quad 1 \le j \le 8, \tag{7.21}$$

to set up an orthonormal basis for $\mathcal{V}$ corresponding to a given one in $\mathcal{E}_0^*$.

Furthermore, in [**2**] it was shown that the first three equations in (7.20) mean that the distribution $\mathcal{D}$ is involutive and each of its leaves induces an isometry of M_- that extends, by the last equation of (7.20) which means that the forms on its left hand side annihilate the distribution $\mathcal{D}$, to an ambient isometry so that the isoparametric hypersurface is one of the two constructed by Ferus, Karcher, and Münzner.

Converted to the language of the unit bundle of M_+ at (x, n) instead, where the shape operator S_n has the eigenspaces E_0, E_+, E_-, the first three equations of (7.20) say, in view of (6.1), (6.2), (6.4), (6.5), that there is an orthogonal map Q that identifies the jth basis vector $e_{\mu=j} \in E_-$ with $-e_{\alpha=j} \in E_+$ so that

$$\begin{aligned} &B_a = C_a, \ \forall a, \\ &A_a \text{ is skew-symmetric}, \ \forall a, \\ &A_a^{\#} \text{ is skew-symmetric}, \ \forall a. \end{aligned} \tag{7.22}$$

The first item of (7.22) is true. Indeed (7.17) and (7.18) mean that if we perform orthogonal row operations by U and W we may assume

$$d_1(x) = g_1(x), \quad d_2(x) = g_2(x).$$

That is, if we define the bundle map Q that swaps the first (last) two μ-rows of $g(x)$ in C_a with the last (first) two α-rows of $d(x)$ in B_a and leaves all remaining four rows of B_a and C_a unchanged, then $B_a = C_a$ via the identification Q (i.e., we may assume $d_a = g_a$ via Q).

It suffices to establish the second item of (7.22). Now δ_a is skew-symmetric by Corollary 7.1. $z_a, 1 \le a \le 3$ are skew-symmetric since $z_a, 1 \le a \le 3$, generate the Clifford algebra C_3 by (7.1), while the upper left blocks of $A_a, 4 \le a \le 7$ are zero. The nature of Q does not change the skew-symmetry of these blocks.

Next, with $d_a = g_a$ via Q in place, we derive from (3.19a) and (3.19b) (with $\Delta = 0$) that we have $\beta_a = \gamma_a^{tr}$. However, we can now change the sign of the last four α-rows of A_a without affecting the skew-symmetry of δ_a and the property $d_a = g_a, c_a = f_a$, so that now

$$\beta_a = -\gamma_a^{tr}, \quad 1 \le a \le 7.$$

That is, A_a is now skew-symmetric for all $1 \le a \le 7$ with this modified Q.

It remains to establish the last item of (7.20), knowing that the first three equations are true via Q. By [**3**, Lemma 2, p. 11], the last item holds true if either $\alpha = i$ or $\alpha = j$ indexes a basis vector in the image of the linear map

$$H : \mathcal{E}_+^* \oplus \mathcal{E}_-^* \to \mathcal{E}_0^*, \quad (e_a, e_p) \mapsto \sum_\alpha S_{\alpha p}^a e_\alpha, \tag{7.23}$$

which is easily seen to be the direct sum of all $e_{\alpha=l}$ for $l \neq 3, 4$ (i.e., the 3rd and 4th rows of B_a are zero for all $1 \leq a \leq 7$). Thus, it suffices to show that the last item of (7.20) is valid for $i = 3, j = 4$ in the α-range.

The left hand side of the last equation in (7.20) annihilates the vectors in $\mathcal{E}_+^* \oplus \mathcal{E}_-^* \subset \mathcal{D}$ because they are horizontal, so that, as said below (7.19), θ_4^3 and $\theta_{4'}^{3'}$ annihilate them since exactly 3 and 4 (respectively, $3'$ and $4'$) are in the same α (respectively, μ) range. (It is understood that by 3 we mean $\alpha = 3$ and by $3'$ we mean $\mu = 3$, etc.)

We show the left hand side of the last equation in (7.20) annihilates $\mathcal{F} \subset \mathcal{D}$ as well. For

$$v := e_{l'} - e_l \in \mathcal{F},$$

we calculate

$$\theta_4^3(v) = -\theta_4^3(e_l), \quad \theta_{4'}^{3'}(v) = \theta_{4'}^{3'}(e_{l'}) \tag{7.24}$$

again by what is said below (7.19).

Since the calculation is pointwise, we first look at the geometry before we proceed. For $x \in M_+$ and n in the unit normal sphere to M_+ at x, the map

$$f : (x, n) \mapsto (x^*, n^*) = ((x + n)/\sqrt{2}, (x - n)/\sqrt{2}) \tag{7.25}$$

sets up a diffeomorphism between the normal bundles of M_+ and M_-. Fix a point (x_0, n_0) in the unit normal bundle of M_+, consider the two sets

$$S_+ := \{(x, n) : x + n = x_0 + n_0\}, \quad S_- := \{(x, n) : x - n = x_0 - n_0\}.$$

$S_\pm$ are two 8-dimensional spheres. Indeed, taking derivative of $x \pm n = c$ with c a constant, we have $dx \pm dn = 0$, which means that a typical tangent space to $S_\pm$ is the eigenspace $\mathcal{E}_\pm$ at (x, n), respectively.

The diffeomorphism f maps S_+ to a sphere whose tangent space at (x_0^*, n_0^*) is $\mathcal{V}$, so that it is the fiber of the unit normal bundle of M_- over x_0^*, and f maps S_- to a sphere whose tangent space at (x_0^*, n_0^*) is the horizontal $\mathcal{E}_0^*$. Thus to calculate the quantities in (7.24), it suffices to observe that (7.23) gives us the information

$$\dim\Big(\bigcap_{a=1}^{7} \mathrm{kernel}(B_a^{tr})\Big) = 2.$$

This translates to S_+ to say that the tangent space to S_+ at (x, n) is identified with E_+ of the second fundamental form S_n, in which there naturally sits a 2-dimensional plane that is the intersection of all kernels of the B^{tr}_m-block of S_m with m perpendicular to n at x, which form a 2-plane bundle $\mathcal{P}_+$ over S_+. By the same token there is a 2-plane bundle $\mathcal{P}_-$ over S_- which comes from the intersection of all kernels of the C^{tr}_m-block of S_m with m perpendicular to n at x. Now, the above fact that after swapping rows we may assume $d_a = g_a, 1 \leq a \leq 7$, means that once we set up the coordinate system of the ambient Euclidean space by the eigenspace decomposition

$$\mathbb{R}x \oplus \mathbb{R}n \oplus E_0 \oplus E_+ \oplus E_-$$

of the shape operator S_n at x for $(x, n) \in S_+$, where the third and fourth rows of B_a are zero for all $1 \leq a \leq 7$, we may assume, after swapping the third and fourth rows with the first and second, that $\mathcal{P}_+$ and $\mathcal{P}_-$ are parametrized identically in the coordinates. That is, in the coordinates we can parametrize S_+ and S_- via an isometry ι in which $\mathcal{P}_+$ is brought to $\mathcal{P}_-$. As a consequence, via the diffeomorphism f in (7.25), a local basis (e_3, e_4) spanning $\mathcal{P}_+$ is converted to one on the image sphere whose tangent space at (x^*_0, n^*_0) is $\mathcal{V}$, and local basis (e'_3, e'_4) spanning $\mathcal{P}_-$ is converted to one on the image sphere whose tangent space at (x^*_0, n^*_0) is $\mathcal{E}^*_0$. Thus through the isometry ι we see that

$$\theta^3_4 = \langle de_3, e_4 \rangle = \langle de'_3, e'_4 \rangle = \theta^{3'}_{4'},$$

which gives (7.24), remarking that there the extra sign is a result of the sign convention in our identification map Q in (7.21), whose choice is in agreement with that of an isoparametriic hypersurface constructed by Ferus, Karcher, and Münzner.

The four equations in (7.20) are satisfied. Thus the isoparametric hypersurface is one of the two constructed by Ferus, Karcher, and Münzner, if it is not the one constructed by Ozeki and Takeuchi.

q.e.d.

APPENDIX I

We give certain codimension 2 estimates needed for imposing constraints on 1-, 2-, and 3-nullity in Section 4.

Lemma .7. *Consider $\mathbb{C}^{15} = \mathbb{C}^8 \oplus \mathbb{C}^7$ parametrized by (x, z). Consider the homogeneous equations of degree 2*

$$f_0 := \sum_{\alpha=1}^{8} (x_\alpha)^2 = 0, \quad f_i := \sum_{\alpha=1,p=1}^{8,7} \theta^i_{\alpha p} x_\alpha z_p = 0, \quad i = 1, 2, 3.$$

Let Z_k be the variety carved out by $0 = f_0 = \cdots = f_k, 0 \leq k \leq 3$. Suppose f_1, f_2, f_3 are linearly independent. Then $Z_k, 0 \leq k \leq 3$, are irreducible of codimension $k + 1$. For an f_4 of homogeneous degree 2 *linearly independent from f_0, f_1, f_2, f_3, we have that f_0, f_1, f_2, f_3, f_4 form a regular sequence and so they carve out a subvariety of codimension* 5.

Proof. The singular set of f_0 consists of points of the form $(0, z)$. Hence the codimension 2 estimate goes through for Z_0. Set

$$R_0 := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad R_k := \begin{pmatrix} 0 & \theta_k \\ \theta_k^{tr} & 0 \end{pmatrix}, k = 1, 2, 3,$$

where the identity matrix is of size 8-by-8 and θ_k is the 8-by-7 matrix whose entries are $\theta^k_{\alpha p}$. As in (3.1), we estimate the dimension of the kernel of

$$S := c_0 R_0 + \cdots + c_k R_k$$

with $[c_0; \cdots : c_k] \in \mathbb{C}P^k, k = 1, 2, 3$. For simplicity, we may assume $c_0 = 1$. Then

$$S := \begin{pmatrix} I & \Theta_k := \sum_{l=1}^{k} c_l \theta_l \\ (\Theta_k)^{tr} & 0 \end{pmatrix},$$

whose kernel elements $(x, z)^{tr}$ satisfies

$$x + \Theta_k z = 0, \quad (\Theta_k)^{tr} x = 0.$$

From this we see that

$$(\Theta_k)^{tr} \Theta_k z = 0,$$

so that the dimension of z is at most 6 for a generic choice of $[c_0 : \cdots : c_k]$ (respectively, 7 for a nongeneric choice) because the independence of p_1, p_2, p_3 dictates that Θ_k is nonzero for such a generic choice. Therefore, the fact that $x = -\Theta_k z$ implies that the kernel dimension is at most 6 for a generic parameter $[c_0 : \cdots : c_k]$ of dimension k. Hence the total dimension is at most $6 + k$ (respectively, $7 + (k - 1) = 6 + k$). On the other hand, $\dim(Z_k) - 2 \geq (15 - k - 1) - 2 = 12 - k$. Therefore, similar arguments leading to (3.12) ensure that the codimension 2 estimate goes through for $Z_k, 1 \leq k \leq 3$. q.e.d.

Lemma .8. *Consider $\mathbb{C}^{14} \simeq \mathbb{C}^7 \oplus \mathbb{C}^7$ parametrized by (x, z), and consider the homogeneous equations of degree* 2

$$f_0 := \sum_{\alpha=1}^{7} (x_\alpha)^2 = 0, \quad f_i := \sum_{\alpha=1, p=1}^{7,7} \theta^i_{\alpha p} x_\alpha z_p + z_7 z_p \text{ terms } = 0$$

for $i = 1, 2$. Let Z_k be the variety carved out by $0 = f_0 = \cdots = f_k, 0 \leq k \leq 2$. Suppose $\sum_{\alpha=1,p=1}^{7,6} \theta^i_{\alpha p} x_\alpha z_p, i = 1, 2$, are linearly independent. Then $Z_k, 0 \leq k \leq 2$, are irreducible of codimension $k + 1$. For an f_3 of homogeneous degree 2 linearly independent from f_0, f_1, f_2, we have that f_0, f_1, f_2, f_3 form a regular sequence and so they carve out a subvariety of codimension 4.

Proof. The singular set of f_0 consists of points of the form $(0, z)$. Hence the codimension 2 estimate goes through for Z_0. Set

$$R_0 := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad R_k := \begin{pmatrix} 0 & \theta_k \\ \theta_k^{tr} & \tau_k \end{pmatrix}, \quad k = 1, 2,$$

where I is 7-by-7, the 7-by-7 θ_k is defined similarly as in the preceding lemma, and τ_k is a 7 by 7 symmetric matrix whose only nonzero row and column are the last one corresponding to the coefficients of the $z_7 z_p$ terms of f_k. Again we estimate the dimension of the kernel of

$$S := c_0 R_0 + \cdots + c_k R_k$$

with $[c_0; \cdots : c_k] \in \mathbb{C}P^k, k = 1, 2, 3$. For simplicity, we may assume $c_0 = 1$. Then

$$S := \begin{pmatrix} I & \Theta_k := \sum_{l=1}^k c_l \theta_l \\ (\Theta_k)^{tr} & \Pi_k := \sum_l c_l \tau_l \end{pmatrix},$$

whose kernel elements $(x, z)^{tr}$ satisfies

$$x + \Theta_k z = 0, \quad (\Theta_k)^{tr} x + \Pi_k z = 0.$$

From this we see that

$$((\Theta_k)^{tr} \Theta_k + \Pi_k) z = 0,$$

so that the dimension of z is at most 6 for a generic choice of $[c_0 : \cdots : c_k]$ (respectively, 7 for a nongeneric choice) because the independence of $\sum_{\alpha=1,p=1}^{7,6} \theta_{\alpha p}^i x_\alpha z_p, i = 1, 2$, dictates that the upper left 6-by-6 block of $(\Theta_k)^{tr} \Theta_k$ is nonzero for such a generic choice. Therefore, the fact that $x = -\Theta_k z$ implies that the kernel dimension is at most 6 for a generic parameter $[c_0 : \cdots : c_k]$ of dimension k. Hence the total dimension is at most $6 + k$ (respectively, $7 + (k - 1) = 6 + k$). On the other hand, $\dim(Z_k) - 2 \geq (14 - k - 1) - 2 = 11 - k$. Therefore, the codimension 2 estimate goes through for $Z_k, 0 \leq k \leq 2$. q.e.d.

Lemma .9. *By the same token, if over* $\mathbb{C}^{13} = \mathbb{C}^6 \oplus \mathbb{C}^7$ *we are given*

$$f_0 := \sum_{\alpha=1}^{6} (x_\alpha)^2 = 0, \quad f_i := \sum_{\alpha=1,p=1}^{6,7} \theta_{\alpha p}^i x_\alpha z_p + z_6 z_p \text{ terms} + z_7 z_p \text{ terms } = 0,$$

$1 \leq i \leq 2$. *Let* Z_k *be the variety carved out by* $0 = f_0 = \cdots = f_k, 0 \leq k \leq 2$. *Suppose* $\sum_{\alpha=1,p=1}^{6,5} \theta_{\alpha p}^i x_\alpha z_p, i = 1, 2$, *are linearly independent, then the codimension 2 estimate goes through for* $k \leq 2$, *and so* $Z_k, k \leq 2$, *are irreducible of codimension* $k + 1$. *For an* f_3 *of homogeneous degree 2 linearly independent from* f_0, f_1, f_2, *we have that* f_0, f_1, f_2, f_3 *form a regular sequence and so they carve out a subvariety of codimension* 4.

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