TAUT SUBMANIFOLDS ARE ALGEBRAIC

QUO-SHIN CHI

ABSTRACT. We prove that every (compact) taut submanifold in Euclidean space is real algebraic, i.e., is a connected component of a real irreducible algebraic variety in the same ambient space.

1. Introduction

An embedding f of a compact, connected manifold M into Euclidean space $\mathbb{R}^n$ is *taut* if every nondegenerate (Morse) Euclidean distance function,

$$L_p : M \to \mathbb{R}, \quad L_p(z) = d(f(z), p)^2, \quad p \in \mathbb{R}^n,$$

has $\beta(M, \mathbb{Z}_2)$ critical points on M, where $\beta(M, \mathbb{Z}_2)$ is the sum of the $\mathbb{Z}_2$-Betti numbers of M. That is, L_p is a perfect Morse function on M.

A slight variation of Kuiper's observation in [7] gives that tautness can be rephrased by the property that

$$H_j(M \cap B, \mathbb{Z}_2) \to H_j(M, \mathbb{Z}_2) \tag{1.1}$$

is injective for all closed disks $B \subset \mathbb{R}^n$ and all $0 \le j \le \dim(M)$. As a result, tautness is a conformal invariant, so that via stereographic projection we can reformulate the notion of tautness in the sphere S^n using the spherical distance functions. Another immediate consequence is that if $B_1 \subset B_2$, then

$$H_j(M \cap B_1) \to H_j(M \cap B_2) \tag{1.2}$$

is injective for all j.

Kuiper in [8] raised the question whether all taut submanifolds in $\mathbb{R}^n$ are real algebraic. We established in [4] that a taut submanifold in $\mathbb{R}^n$ is real algebraic in the sense that, it is a connected component of a real irreducible algebraic variety in the same ambient space, provided the submanifold is of dimension no greater than 4.

2000 *Mathematics Subject Classification.* Primary 53C40.

Key words and phrases. Dupin hypersurface, taut submanifold, semi-algebraic set.

The author was partially supported by NSF Grant No. DMS-0103838.

In this paper, we prove that all taut submanifolds in $\mathbb{R}^n$ are real algebraic in the above sense, so that each is a connected component of a real irreducible algebraic variety in the same ambient space. In particular, any taut hypersurface in $\mathbb{R}^n$ is described as $p(t) = 0$ by a single irreducible polynomial $p(t)$ over $\mathbb{R}^n$. Moreover, since a tube with a small radius of a taut submanifold in $\mathbb{R}^n$ is a taut hypersurface [10], which recovers the taut submanifold along its normals (we will see this in (2.4) below), understanding a taut submanifold, in principle, comes down to understanding the hypersurface case defined by a single algebraic equation.

To achieve the goal, on the one hand we continue to explore the property that certain multiplicity sets are of finite ends as studied in [4]. On the other we employ Morse-Bott theory [3] and further real algebraic geometry in conjunction with Ozawa's theorem [9] to obtain, in the hypersurface case, a fine structure of the set where the principal multiplicities are not locally constant. As a byproduct, the crucial local finiteness property that is decisive in [4] for establishing that a taut submanifold is algebraic falls out.

It is more convenient to prove that a taut submanifolds in the sphere is real algebraic, though occasionally we will switch back to Euclidean space when it is more convenient for the argument. Since a spherical distance function $d_p(q) = \cos^{-1}(p \cdot q)$ has the same critical points as the Euclidean height function $\ell_p(q) = p \cdot q$, for $p, q \in S^n$, a compact submanifold $M \subset S^n$ is taut if and only if it is *tight*, i.e., every nondegenerate height function ℓ_p has the total Betti number $\beta(M, \mathbb{Z}_2)$ of critical points on M. We will use both d_p and ℓ_p interchangeably, whichever is more convenient for our argument.

Our proof is based on a fundamental result on taut submanifolds due to Ozawa [9].

Theorem 1 (Ozawa). *Let M be a taut submanifold in S^n, and let $\ell_p, p \in S^n$, be a linear height function on M. Let $x \in M$ be a critical point of ℓ_p, and let S be the connected component of the critical set of ℓ_p that contains x. Then S is*
(a) *a smooth compact manifold of dimension equal to the nullity of the Hessian of ℓ_p at x;*
(b) *nondegenerate as a critical manifold;*
(c) *taut in S^n.*

In particular, ℓ_p is perfect Morse-Bott [3]. We call such a connected component of a critical set of ℓ_p a *critical submanifold* of ℓ_p.

An important consequence of Ozawa's theorem is the following [5].

Corollary 2. *Let M be a taut submanifold in S^n. Then given any principal space T of any shape operator S_ζ at any point $x \in M$, there exists a submanifold S (called a curvature surface) through x whose tangent space at x is T. That is, M is Dupin* [10].

Let us remark on a few important points in the corollary. It is convenient to work in the ambient Euclidean space $\mathbb{R}^n$. Let μ be the principal value associated with T. Consider the focal point $p = x+\zeta/\mu$. Then the critical submanifold S of the (Euclidean) distance function L_p through x is exactly the desired curvature surface through x. The unit vector field

$$\zeta(y) := \mu(p - y) \tag{1.3}$$

for $y \in S$ extends ζ at x and is normal to and parallel along S. The $(n-1)$-sphere of radius $1/\mu$ centered at p is called the *curvature sphere* of Z.

2. The proof

We do an inductive argument on the following statement:

$\mathcal{S}(n)$: All taut submanifolds in S^n are real algebraic.

The statement is true for $n = 1$ since a 0-dimensional taut submanifold is a point. Assuming the statement is true for all $k \leq n-1$.

We first handle the case when M is a hypersurface. Fix a unit normal field $\mathbf{n}$ over M once and for all. We label the principal curvatures of M by $\lambda_1 \leq \cdots \leq \lambda_{n-1}$, which are Lipschitz-continuous functions on M because the principal curvature functions on the linear space $\mathcal{L}$ of all symmetric matrices are Lipschitz-continuous by general matrix theory [1, p. 64], and the Hessian of of M is a smooth function from M into $\mathcal{L}$. Let $\lambda_j = \cot(t_j)$ for $0 < t_j < \pi$. We have the Lipschitz-continuous focal maps

$$f_j(x) = \cos(t_j)x + \sin(t_j)\mathbf{n}.$$

In fact, the lth focal point $f_l(x)$ along $\mathbf{n}$ emanating from x is antipodally symmetric to the $(n-l)$th focal point along $-\mathbf{n}$ emanating from x. The spherical distance functions $d_{f_l(x)}$ tracing backward following $-\mathbf{n}$ thus assumes the same critical point x as the distance function $d_{-f_l(x)}$ tracing backward following $\mathbf{n}$; thus we may just consider the former case without loss of generality. Accordingly, we refer to a focal point p as being $f_j(x)$ for some x and j.

By the inductive hypothesis, Z must be algebraic since Z lies in its curvature sphere by Corollary 2.

As mentioned earlier, we can regard $M \subset S^n$ as being tight. Suppose Z is a critical submanifold of M cut out by the height function ℓ_p; assume $\ell_p(Z) = 0$ without loss of generality. Let $W \subset M$ be a tubular neighborhood of Z so small that $\ell_p^{-1}(0)$ is the only critical set of ℓ_p in W. (We will call such a W a *neck* around Z.)

Let us slightly perturb ℓ_p by a linear function g with small coefficients such that g is not a multiple of ℓ_p (otherwise $\ell_p + g$ is just ℓ_p in essence). Then $\ell_p + g = \ell_q$ for some q close to p. Z is not a critical submanifold of $\ell_p + g$, or equivalently, of g since $q \neq p$.

Since Z is taut by Ozawa's theorem, in general the height function g cuts Z in several critical submanifolds $Z_1, \cdots, Z_l$; without loss of generality, we assume these critical submanifolds of Z correspond to different critical values of g. It suffices to consider Z_1, for instance. Assume the codimension of Z_1 in Z is t and the dimension of Z is s. Let us parametrize W by $v_1, \cdots, v_t, v_{t+1}, \cdots, v_s, u_1, \cdots, u_{n-1-s}$ around 0, where $v_{t+1} \cdots, v_s$ parametrize Z_1, $v_1, \cdots, v_s$ parametrize a neck around Z_1 in Z, and lastly the variables $v_1, \cdots, v_s, u_1, \cdots, u_{n-1-s}$ parametrize the neck W of dimension $n-1$, which is the dimension of M, around Z. It is understood that 0 in the coordinate system corresponds to a point on Z_1. As in [9], we can assume

$$\ell_p = \sum_{j=1}^{n-1-s} \alpha_j u_j^2 + O(3),$$

$$g = h(u) + \sum_{i=1}^{t} \beta_i v_i^2 + O(3),$$

with

$$h(u) = \sum_{j=1}^{n-1-s} a_i u_i + \sum_{j,k=1}^{n-1-s} b_{jk} u_j u_k$$

for some small coefficients a_i and b_{jk}, where α_j and β_i are all nonzero constants. Note that the cross uv-terms can always be canceled by an appropriate linear change of coordinates. Moreover, there are no $v_{t+1}, \cdots, v_s$ present in g because when we set the u-variables equal to zero, Z_1 parametrized by $v_{t+1}, \cdots, v_s$ is a critical submanifold of g over Z. Differentiating and setting the derivatives equal to zero, we obtain

$$0 = \partial(\ell_p + g)/\partial u_j = a_j + 2\alpha_j u_j + 2 \sum_{l=1}^{n-1-s} b_{jl} u_l + O(2) := F_j$$

for $1 \leq j \leq n-1-s$, and

$$0 = \partial(\ell_p + g)/\partial v_i = 2\beta_i v_i + O(2) := G_i$$

for $1 \leq i \leq t$.

Since a_i and b_{jk} are small quantities, we know

$$\partial(F_1, \cdots, F_{k-s})/\partial(u_1, \cdots, u_{n-1-s}) \neq 0$$

at $u_1 = \cdots = u_{n-1-s} = 0$. Therefore, the implicit function theorem implies that $u_1, \cdots, u_{n-1-s}$ are all functions of $v_1, \cdots, v_s$. Likewise, since all β_i are nonzero, we can in turn solve $v_1, \cdots, v_t$ in terms of $v_{t+1}, \cdots, v_s$, the coordinates of Z_1. The critical set is thus a graph over Z_1. Hence we have the following.

Proposition 3. *Consider a neck W around a critical submanifold Z that is cut out by ℓ_p in M. Let $N_1, \cdots, N_l$ be necks around the critical submanifolds $Z_1, \cdots, Z_l$ cut out by g in Z, respectively. Set up a finite number of aforementioned coordinate charts and let*

$$\pi : (v_1, \cdots, v_s, u_1, \cdots, u_{n-1-s}) \mapsto (v_1, \cdots, v_s).$$

be the projection. Then the critical set of $\ell_p + g$ in $\pi^{-1}(N_i)$ is a graph over Z_i.

On the other hand, at a point $x \in Z$ away from $Z_1, \cdots, Z_l$, we can still parametrize W around x by $v_1, \cdots, v_s, u_1, \cdots, u_{n-1-s}$ where $v_1, \cdots, v_s$ parametrize Z around x identified with 0. Then slightly different from the earlier expression we have

$$\ell_p = \sum_{j=1}^{n-1-s} \alpha_j u_j^2 + O(3),$$

$$g = h(u) + \sum_{i=1}^{s} \gamma_i v_i + \sum_{i=1}^{s} \delta_i v_i^2 + O(3)$$

where at least one of γ_i is nonzero since p is a nondegenerate point of g on Z. Once more by setting $\partial(\ell_p + g)/\partial u_j$ equal to zero we see that $u_1, \cdots, u_{n-1-s}$ are all functions of $v_1, \cdots, v_s$. On the other hand, we may assume none of the δ_i are zero. For, suppose $\gamma_1 \neq 0$ and some $\delta_j = 0$. Then replacing v_1 by $v_1 + v_j^2$ and keeping all other variables unchanged will result in a nonzero coefficient for v_j^2 with all other δ_i unchanged. Then setting $\partial(\ell_p + g)/\partial v_i$ equal to zero yields

$$0 = \gamma_i + 2\delta_i v_i + O(2) =: H_i, \quad 1 \leq i \leq s.$$

As before, we see

$$\partial(H_1, \cdots, H_s)/\partial(v_1, \cdots, v_s) \neq 0.$$

Therefore, the implicit function theorem implies that there is only a single point solution, which is a nondegenerate critical point of $\ell_p + g$ in a small neighborhood of x in W, which we can thus ignore.

Recall the local finiteness property in [4] that holds the key for proving that a taut hypersurface is real algebraic. We denote by $\mathcal{G}$ the subset of M where the multiplicities of principal values are locally constant, and by $\mathcal{G}^c$ its complement in M.

Definition 4. A connected Dupin hypersurface M of S^n has the *local finiteness property* if there is a subset $S \subset \mathcal{G}^c$, closed in M, such that S disconnects M into only a finite number of connected components, and for each point $x \in \mathcal{G}^c \setminus S$, there is an open neighborhood O of x in M such that $O \cap \mathcal{G}$ contains a finite number of connected open sets whose union is dense in O.

It suffices to establish that $\mathcal{G}$ satisfies the local finiteness property for M to be real algebraic [4, Theorem 8]. We begin with a convenient lemma. Recall the global minimum or maximum level set of a height function on M is called a top set. Setting $j = 0$ in (1.1) we see a top set is always connected.

Lemma 5. *Let T_i be a sequence of top sets of dimension l at q_i in the taut hypersurface M. Suppose T_i converge to a top set T of dimension m at p. Then $H_l(T) \neq 0$.*

Proof. First off, the top-dimensional homology of a top set of M is nonzero. This follows from the Poincare duality (with $\mathbb{Z}_2$ coefficients) and that a top set is connected.

Now let W be a tubular neighborhood of the top set T at p so small that T is the only critical set in it. Let j be so large that a tubular neighborhood W_j of the top set T_j, containing only T_j, is brought to lie inside W. Then by (1.2)

$$H_k(T_j) \to H_k(T)$$

is an injection for all k. It follows that $H_l(T)$ is nonzero by what is said in the preceding paragraph. □

Returning to establishing the local finiteness property, let $S \subset \mathcal{G}^c$ be the set of points where the principal multiplicities are $(1, \dim(M) - 1)$ or $(\dim(M) - 1, 1)$. The set is closed; or else a boundary point of which would assume the single principal multiplicity $(\dim(M))$ so that M would be a sphere. S must be a subset of $\mathcal{G}^c$. This is because if multiplicities $(1, \dim(M) - 1)$ exist on an connected open set $O \subset \mathcal{G}$, let $p_i \in O$ be a sequence which converges to p on the boundary of O. The multiplicities at p must remain to be $(1, \dim(M))$, or else it would drop to the single multiplicity $(\dim(M))$. On the other hand, there must be a sequence q_i of points converging to p with fixed multiplicities $(\cdots, l)$ where $l < \dim(M) - 1$. Therefore, on the one hand, the curvature surface S_i at p_i with principal multiplicity $\dim(M) - 1$, which is a top set sphere of dimension $\dim(M) - 1$, converges to the top set curvature surface at p, which is also a sphere S_p of dimension $\dim(M) - 1$. This is because each S_i is cut out from its curvature sphere by a unique hyperplane L_i in the ambient Euclidean space, so that the limiting hyperplane also cuts out a sphere, which is S_p, from the limiting curvature sphere. On the other hand, at q_i the curvature surface T_i with principal multiplicity l is a top set as well, and so by Lemma 5 the l-dimensional homology in S_p is nontrivial, which is absurd.

We next show that S disconnects M into only finitely many components. Recall the following definition in [4].

Definition 6. For each natural number m we define $(U_m^*)^+$ to be the collection of all $x \in M$ for which there is a $t > 0$ such that (x, t) is a regular point of the normal exponential map

$$E : (x, t) \mapsto \cos(t)x + \sin(t)\mathbf{n}$$

and such that the spherical distance function d_y, where $y = E(x, t)$, has index m at x.

We showed in Corollary 20 of [4] that $(U_m^*)^+$ has a finite number of connected components for all m.

Remark 7. The $+$ sign in $(U_m^*)^+$ is merely to indicate that we traverse in the positive $\mathbf{n}$ direction, which we have agreed to undertake earlier.

Consider $A_m := (U_m^*)^+$ for $m = 1, \cdots, \dim(M) - 1$. Let $B := \cup_{m=2}^{\dim(M)-1} A_m$ and $A := A_1$. Then it is readily checked that $M = A \cup B$ and furthermore $C := A \cap B$ is exactly A with points of multiplicities $(1, \dim(M) - 1)$ removed. Therefore, the Mayer-Vietoris sequence

$$0 \to H^0(M) \to H^0(A) \oplus H^0(B) \to H^0(C) \to H^1(M) \to$$

establishes that C has finitely many components, which is what we are after.

Now let $x \in \mathcal{G}^c \setminus S$ and let Z through x be a critical submanifold with focal point p. By the nature of S we know that

$$\dim(Z) \leq \dim(M) - 2;$$

in particular, Z does not disconnect M. From this point onward we diversify into two cases.

Case 1. None of the curvature spheres of $\ell_p + g$ contain Z.

This means that Z is not a level set of g so that g cuts Z in proper taut submanifolds. Let I be the index range such that

$$p = f_a(x), \forall a \in I. \tag{2.1}$$

Let W be a neck of Z. Let $O \subset W$ around x be an open ball. The set

$$\mathcal{F}_O := \cup_{a\in I} f_a(O)$$

is a connected set of focal points around the focal point p.

We pick the open ball O so small that any critical submanifold of $\ell_p + g = \ell_q$, for focal points $q \in \mathcal{F}_O$, lies completely in W when its intersection with O is not empty. (From now on we identify an element q in $\mathcal{F}_O$ with the corresponding g interchangeably.) Proposition 3 ensures that these critical submanifolds of ℓ_q on W are all graphs over the corresponding critical submanifolds Z_g that g cut out on Z.

Consider the incidence space $\mathcal{I} \subset \mathcal{F}_O \times W \subset S^n \times S^n$ given by

$$\begin{aligned}\mathcal{I} := \{(g, z) : z \in \text{ a critical submanifolds of}\, \ell_p + g \,\text{in}\, W,\\ \text{and}\dim(Z_h)\,\text{is not locally constant for}\, h\, \text{around}\, g\}.\end{aligned}$$

Let

$$\Pi : S^n \times S^n \to S^n$$

be the standard projection onto the second factor. Then

$$W \cap (\mathcal{G}^c \setminus Z) = \Pi(\mathcal{I}). \tag{2.2}$$

Note that $\Pi|_{\mathcal{I}}$ is an open finite (hence proper) map; the finiteness is because through each point in M there are only at most $\dim(M)$ worth of critical submanifolds, while the openness follows from that of Π.

The following lemma, based on our inductive hypothesis, makes the structure of $\mathcal{I}$ clear.

Lemma 8. *$\mathcal{I}$ is a piecewise smooth simplicial complex of dimension at most $\dim(M) - 1$.*

Proof. Since $Z \subset S^n$ is algebraic by the inductive hypothesis, the set UN^o of unit normals ξ of Z at which the shape operator S_ξ has multiplicity change is semialgebraic. This can be seen as follows. Let $\dim(M) = s$ and let $(y, \zeta) \in Z \times S^{n-s-1}$ parametrize the unit normal bundle of Z. The characteristic polynomial of S_ξ is of the form

$$\lambda^s + a_{s-1}\lambda^{s-1} + \cdots + a_1\lambda + a_0,$$

where $a_1, \cdots, a_{s-1}$ are polynomials in the zero jet of ζ and the second jets of y; hence they are Nash functions. By the slicing theorem [2, p. 30], $Z \times S^{n-l-1}$ is decomposed into finitely many disjoint semialgebraic sets $A_1, \cdots, A_m$, where each A_i is equipped with semialgebraic functions $\eta_{i_1} < \cdots < \eta_{i_{l_i}}$ that solve the characteristic polynomial. Where multiplicities are not locally constant occurs at some $A_1, \cdots, A_m$ whose dimensions are lower than $n-1$, the dimension of $Z \times S^{n-1-s}$.

Now in view of Corollary 2, for a unit normal ξ to Z, we let $q_\xi^1, q_\xi^2, \cdots,$ and $q_\xi^{\dim(Z)}$ be the focal point of the curvature surface through the base point of ξ corresponding to the principal curvature function $\lambda^1(\xi), \cdots,$ and $\lambda^{\dim(Z)}(\xi)$ of S_ξ, respectively. The remark following Corollary 2 gives the focal maps $g^1, \cdots, g^{\dim(Z)}$ that send ξ to the respective focal points; by the algebraic nature of Z, all these maps are semialgebraic. Consider the semialgebraic set $\mathcal{X} \subset UN^o \times S^n \times S^n$ defined by

$$\mathcal{X} := \{(\xi, q, r) : q = g^j(\xi) \text{ for some } j; r \text{ belongs a critical set of } Z \text{ of the height function } \ell_q \text{ centered at } q\}.$$

Due to the nature of all these defining functions, $\mathcal{X}$ is semialgebraic. (For instance, critical submanifolds are obtained by setting the first derivative of the height function equal to zero on Z, which is a semialgebraic process.) Let $pr : UN^o \times S^n \times S^n \to S^n \times S^n$ be the standard projection, and let $\mathcal{J} := pr(\mathcal{X})$. The set $\mathcal{J}$ is also semialgebraic.

We now estimate the dimension of $\mathcal{J}$. Consider the the map

$$PR := \Pi|_{\mathcal{J}}.$$

It is readily seen that $PR : \mathcal{J} \to Z$. For a fixed z in the image of PR, the preimage $PR^{-1}(z)$ consists of the focal points that come from the $\xi \in UN^o$ where the base point of ξ is z. At z, the eigenvalue problem is an algebraic one; therefore, the set S of ξ based at z where principal multiplicities is not locally constant is a subvariety of the unit normal sphere at z of dimension at most $n - \dim(Z) - 2$. Each ξ in S gives rise to at most $\dim(Z)$ worth of taut submanifolds through z, and vice versa, whose focal points are the ones in $PR^{-1}(z)$. Therefore,

$$\dim(PR^{-1}(z)) \le n - \dim(Z) - 2.$$

As a result, as z varies in Z

$$\dim(\mathcal{J}) \le n - \dim(Z) - 2 + \dim(Z) = \dim(M) - 1.$$

Since a semialgebraic set assumes a triangulation of semialgebraic simplicial complexes [2, p. 217], the structure of $\mathcal{J}$ is clear. Consider the map $F : \mathcal{I} \to \mathcal{J}$ given by

$$F : (g, z) \to (g, \pi(z)),$$

where π is given in Proposition 3. The preimage of each point is finite with cardinality at most $\beta(M, \mathbb{Z}_2)$ between the two spaces with the naturally induced metrics. Hence, F is a finite covering map, since for a fixed g the map π maps a critical manifold of $\ell_p + g$ to Z_g diffeomorphically. As a consequence $\mathcal{I}$ inherits from $\mathcal{J}$ a piecewise smooth triangulation of dimension $\dim(M) - 1$ sitting in $S^n \times S^n$. In fact we can work our way down the skeletons of the simplicial complex dimension by dimension. Each open face of the skeleton is defined by a finite set of polynomial functions $H < 0$, so that the pullback maps $H \circ F < 0$ define the corresponding open face for $\mathcal{I}$. □

Since the natural projection $\Pi : S^n \times S^n \to S^n$ into the second slot is an open finite map when restricted to $\mathcal{I}$ as mentioned earlier, we see that at $x \in Z$ with preimages $x_1, \cdots, x_k \in \mathcal{I}$, the projection Π sends k disjoint piecewise smooth (local) finite simplicial complexes $\mathcal{C}_1, \cdots, \mathcal{C}_k$ (of dimension at most $\dim(M) - 1$) around $x_1, \cdots, x_k$, respectively, to $x \in S^n$. Over each $\mathcal{C}_j$, the differential $d\Pi$ is not defined over the skeletons of dimension $\leq \dim(M) - 2$; call this set $\mathcal{K}_j$, which is a rectifiable set [6, p. 251]. Hence by the general area-coarea formula [6, p. 258]

$$\mathcal{H}.\dim(\Pi(\mathcal{K}_j)) \leq \dim(M) - 2$$

since $\Pi_{\mathcal{I}}$ is a finite map; here $\mathcal{H}.\dim$ denotes the Hausdorff dimension. On the other hand, $d\Pi$ is defined over the $(\dim(M) - 1)$-dimensional open faces $\mathcal{F}_{jl}$ of $\mathcal{C}_j$. By Federer's version of Sard's theorem [6, p. 316], the critical value set Θ_{jl} of Π over $\mathcal{F}_{jl}$ satisfies

$$\mathcal{H}^{\dim(M)-1}(\Theta_{jl}) = 0,$$

where $\mathcal{H}^{\nu}$ denotes the Hausdorff ν-dimensional measure. Therefore,

$$\mathcal{H}^{\dim(M)-1}(\Pi(K_j \cup_l \mathcal{F}_{jl})) = 0,$$

which implies that $\Pi(\mathcal{K}_j \cup_l \mathcal{F}_{jl})$ does not disconnect M [11, p. 269]. Over the regular points of $\mathcal{F}_{jl}$ the map $\Pi|_{\mathcal{I}}$ is a finite covering map. As a consequence Π maps $\mathcal{I}$ to k "folded" local complexes around x, each locally disconnects M in only finitely many components.

Case 2. There are some g such that $\ell_p + g$ contain Z.

This means Z is contained in a level set for such g. Suppose Z is contained in a critical submanifold of g. Then by Corollary 2, the height functions ℓ_p and $\ell_p + g = \ell_q$ share the same center of the curvature sphere through Z, so that it must be that $p = q$, which is not the case. Therefore, all points of Z are regular points of g. Similar to the equations following Proposition 3 we have

$$\ell_p = \sum_{j=1}^{n-1-s} \alpha_j u_j^2 + O(3), \tag{2.3}$$
$$g = h(u) + O(3);$$

That g has no v terms is because $g(Z)$ is a constant. Analogous analysis as before shows that $u_1, \cdots, u_{n-1-s}$ are functions of $v_1, \cdots, v_s$, so that the critical manifolds of $\ell + g$ are graphs over Z.

In fact, we can understand all these g explicitly. Let S^l be the smallest sphere containing Z. It is more convenient to view what goes on in R^n when we place the pole of the stereographic projection on the S^l containing Z. Then we are looking at an $\mathbb{R}^l$, which we may assume is the standard one contained in $\mathbb{R}^n$, in which Z sits. Let $E \simeq \mathbb{R}^{n-l}$ be the orthogonal complement of the $\mathbb{R}^l$. Any $\mathbb{R}^{n-l-1}$ in E gives rise to an $\mathbb{R}^{n-1}$ containing Z, and vice versa. Back on the sphere, this means that we have an $(n-l-1)$-parameter family of S^{n-1} containing Z. The focal points of these S^{n-1} is an S^{n-l-1} on the equator. Now the critical sets of this $(n-l-1)$-parameter family of distance functions centered at the focal sphere S^{n-l-1} are all graphs over Z by the analysis following (2.3). It follows that we have a manifold structure $Z \times S^{n-l-1}$ of dimension

$$\dim(Z) + n - l - 1 \leq n - 2 = \dim(M) - 1$$

if $\dim(Z) < l$, in which case, the set of all these critical submanifolds locally disconnects M in at most two components. If on the other hand $\dim(Z) = l$, then $Z = S^l$. The manifold structure $Z \times S^{n-l-1}$ of dimension n then fills up M, which means there is no multiplicity change around Z so that Z can be ignored.

In summary, we have established the local finiteness property, and so M is algebraic when it is a hypersurface.

We now handle the case when M is a taut submanifold. It is more convenient to work in $\mathbb{R}^n$. Let M_ϵ be a tube over M of sufficiently small radius that M_ϵ is an embedded hypersurface in $\mathbb{R}^n$. Then M_ϵ is a taut hypersurface [10], so that by the above M_ϵ is algebraic. Consider the focal map $F_\epsilon : M_\epsilon \to M \subset \mathbb{R}^n$ given by

$$F_\epsilon(x) = x - \epsilon\xi, \tag{2.4}$$

where ξ is the outward field of unit normals to the tube M_ϵ. Any point of M_ϵ has an open neighborhood U parametrized by an analytic algebraic map. The first derivatives of this parametrization are also analytic algebraic [2, p. 54], and thus the Gram-Schmidt process applied to these first derivatives and some constant non-tangential vector produces the vector field ξ and shows that ξ is analytic algebraic on U. Hence F_ϵ is analytic algebraic on U and so the image $F_\epsilon(U) \subset M$ is a semialgebraic subset of $\mathbb{R}^n$. Covering M_ϵ by finitely many sets of this form U, we see that M, being the union of their images under F_ϵ, is a semialgebraic subset of $\mathbb{R}^n$. Then the Zariski closure $\overline{M}^{\text{zar}}$ of M is an irreducible algebraic variety of the same dimension as M and contains M.

The inductive procedure is thus completed.

References

[1] B. Bhata, Matrix Analysis, Graduate Text in Mathematics, Vol. 196, Springer, New York, 1996.

[2] J. Bochnak, M. Coste and M.-F. Roy, Real Algebraic Geometry, vol. 36, Ergebnisse der Mathematik, Springer, Berlin, 1998.

[3] R. Bott, *Nondegenerate critical manifolds*, Ann. Math. **60**(1954), 248-261.

[4] T. Cecil, Q.-S. Chi and G. Jensen, *On Kuiper's question whether taut submanifolds are algebraic*, Pacific J. Math. **234**(2008), 229-247.

[5] T. Cecil, Taut and Dupin submanifolds, pp. 135-180, in Tight and Taut Submanifolds, edited by T. Cecil and S.-S. Chern, Mathematical Sciences Research Institute Publications, vol. 32, 1997.

[6] H. Federer, Geometric Measure Theory, Die Grundlehren der mathematischen Wisenschaften **153**, Springer, New York, 1969.

[7] N. H. Kuiper, *Tight embaddings and maps. Submanifolds of geometrical class three in E^n*, The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979), 97-145, Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[8] N. H. Kuiper, *Geometry in total absolute curvature theory*, pp. 377-392, in Perspectives in Mathematics, edited by W. Jäger et al., Birkhäuser, Basel, 1984.

[9] T. Ozawa, *On critical sets of distance functions to a taut submanifold*, Math. Ann. **276**(1986), 91-96.

[10] U. Pinkall, *Curvature properties of taut submanifolds*, Geom. Dedicata **20**(1986), 79-83.

[11] R. Schoen and S.T. Yau, Lectures on differential geometry, Conference Proceedings and Lecture Notes in Geometry and Topology, I, International Press, Cambridge, MA, 1994.

Department of Mathematics, Campus Box 1146, Washington University, St. Louis, Missouri 63130

E-mail address: chi@math.wustl.edu