The Moduli Space of Branched Superminimal Surfaces of a Fixed Degree, Genus and Conformal Structure in the Four-sphere

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0. Introduction. Minimal immersions from a Riemann surface M into S^n were studied by Calabi ([3]) and Chern ([4]), among many authors. To such an immesion F in S^4, they found a holomorphic quartic form Q_F (to be defined in Section 1) on M. A superminimal immersion is one for which $Q_F = 0$, which is always the case when $M = S^2$. In [2], Bryant studied a superminimal immersion of a higher genus into S^4 by lifting it to $\mathbb{C}P^3$, the twistor space of S^4. The lift of a superminimal immersion is a holomorphic curve, of the same degree as that of the immersion, which is horizontal with respect to the twistorial fibration; more precisely, it is a holomorphic curve in $\mathbb{C}P^3$ satisfying the differential equation

$$z_0dz_1 - z_1dz_0 + z_2dz_3 - z_3dz_2 = 0. \tag{1}$$

Setting $z_0 = 1$, $z_1 + z_2z_3 = f$ and $z_2 = g$, one can solve z_1, z_2, z_3 in terms of the meromorphic functions f and g, which serves as a kind of "Weierstrass representation". Via this representation, Bryant showed the existence of a superminimal immersion from any compact Riemann surface into S^4. However, his existence result does not specify the degree d of the immersion, which is the simplest global invariant of the surface.

In Loo ([12]) and Verdier ([17]), $f_1 = z_1/z_0$ and $f_2 = z_3/z_2$ were chosen in place of the aforementioned f and g. Generically, f_1 and f_2 are of degree d which satisfy $\mathrm{ram}(f_1) = \mathrm{ram}(f_2)$, where $\mathrm{ram}(f)$ denotes the ramification divisor of the meromorphic function f. This gives a scheme of constructing

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the moduli space of all branched superminimal surfaces in S^4 with a fixed degree d and conformal structure. For $M = S^2$, Loo ([12]) showed that the moduli space is connected and has dimension $2d+4$; Verdier ([17]) in addition pointed out that the moduli space has three irreducible components.

In this paper, we propose to carry the investigation over to higher genera. Let $F : M \longrightarrow S^4$ be a superminimal immersion of degree d and let $\tilde{F} : M \longrightarrow \mathbb{C}P^3$ be its horizontal lift (of degree d). Let $L_F = \tilde{F}^*\mathcal{O}(1)$. We may regard $z_0, \cdots, z_3$ above as four sections in $H^0(L_F)$ without common zeros. Define two functions f_1 and f_2 from M to $\mathbb{C}P^1$ by setting $f_1(p) = [z_0(p) : z_1(p)]$ and $f_2(p) = [z_2(p) : z_3(p)]$. Consider now the pairing

$$[[z_0, z_1]] =: z_0 dz_1 - z_1 dz_0, \tag{2}$$

which is an element of $H^0(K \otimes (L_F)^2)$, where K is the canonical bundle of M. Set

$$\mathcal{RAM}\,(f_1) = \text{zero divisor of } [[z_0, z_1]] \,\in\, K \otimes (L_F)^2. \tag{3}$$

$\mathcal{RAM}\,(f_1)$ may be thought of as the "virtual" ramification divisor of f_1 since

$$\mathcal{RAM}(f_1) \;= ram(f_1) + 2B,$$

where B is the divisor of the common zeros of z_0, and z_1. With these, equation (1) merely says $[[z_0, z_1]] = -[[z_2, z_3]]$, and thus $\mathcal{RAM}\,(f_1) = \mathcal{RAM}\,(f_2)$.

Let G^r_d be the space of all r-dimensional linear systems g^r_d of degree d on M, and let W^r_d be the space of all holomorphic line bundles L of degree d such that $\dim H^0(L) \geq r + 1$. A g^1_d determines a holomorphic map from M to $\mathbb{C}P^1$ up to $PGL(2, \mathbb{C})$; the collection of all such maps determined by G^1_d is a principal $PGL(2, \mathbb{C})$-bundle over G^1_d ([14]), to be denoted by R^1_d. Consider the map

$$\mu : R^1_d \longrightarrow G^1_d \longrightarrow W^1_d$$

given by

$$[z_0 : z_1] \overset{\pi_1}{\longmapsto} < z_0, z_1 > \overset{\pi}{\longmapsto} L_F. \tag{4}$$

We will see in Section 2 that the moduli space of horizontal holomorphic curves of degree d of a Riemann surface M, denoted by $\mathcal{M}_d(M)$, is essentially the set of $(f_1, f_2) \in R^1_d \times R^1_d$, where

$$\mathcal{RAM}\,(f_1) = \mathcal{RAM}\,(f_2)$$

and

$$\mu(f_1) = \mu(f_2),$$

with the additional property that $\pi_1(f_1)$ and $\pi_1(f_2)$ have disjoint base loci. (The last condition ensures that the four sections $z_0, \cdots, z_3$ have no common zeros, so that $\tilde{F}$ is of degree d.)

We can now picture $\mathcal{M}_d(M)$ as the set of such pairs (f_1, f_2) sitting over W_d^1, and thus may slice $\mathcal{M}_d(M)$ by $L \in W_d^1$. Let $\mu(f_1) = \mu(f_2) = L$, and let $x = \pi_1(f_1)$ and $y = \pi_1(f_2) \in G_d^1$. Then on the G_d^1 level, each slice is just the collection of pairs (x, y) with $(x, y) \in G(2, H^0(L))$ such that $\mathcal{RAM}\,(x) = \mathcal{RAM}\,(y)$ and x and y have disjoint loci, where $\mathcal{RAM}$ now is the restriction of a projection $\mathcal{R}$ from $\mathbb{P}(\bigwedge^2(H^0(L)))$ to $\mathbb{P}(H^0(K \otimes L^2))$. Notice that if $x = y$, then the branched superminimal immersion constructed out of (f_1, f_2) is totally geodesic. We assume henceforth that $x \neq y$. it follows that x and y generate a sub-Grassmann $G(2,4)$ in $G(2, H^0(L))$. By looking at the singular locus of $\mathcal{RAM}$ restricted to this $G(2,4)$, one sees immediately that one can always continuously deform (x, y) to an element of the form (t, t) for some $t \in G_d^1$; consequently the connectedness of G_d^1 ([1]) enables us to assert the connectedness of $\mathcal{M}_\lceil(\mathcal{M})$ when M is a Riemann surface of genus g with $d > (g+2)/2$. It should be mentioned that the connectedness of $\mathcal{M}_d(M)$ has recently been proved by Guest-Ohnita ([8]) via loop group analysis when the ambient sphere is of arbitrary dimension.

As to the existence of a nontotally geodesic branched superminimal surface of degree d, one must distinguish small degrees from large ones. Notice that the existence of a nontotally geodesic branched superminimal immersion, or rather the existence of the $G(2,4)$ generated by x and y above, implies that $\dim H^0(L) \geq 4$ indeed. Employing this condition and Clifford's Theorem about special divisors on Riemann surfaces, we can show that if $\min(g, 6) \geq d$ the branched superminimal immersions of degree d and genus g are all totally geodesic, except in the case when $d = 6$ and M is hyperelliptic, where $\mathcal{M}_6(M)$ is isomorphic to $\mathcal{M}_3(\mathbb{C}P^1)$. Furthermore, by analyzing all complete linear systems of degree 5 on Riemann surfaces of genus ≤ 4, we are able to conclude that all branched superminimal immersions of degree 5 and genus ≤ 4 are totally geodesic. The upshot of these rsults, which is the context of Theorem 1 in Section 4, is: For $g \geq 1$, all branched superminimal immersions of degree ≤ 5 from any Riemann surface into S^4 are totally geodesic.

When the degree is 6, one readily sees the existence of nontotally geodesic branched superminimal immersions if the Riemann surface is hyperelliptic: Just take a nontotally geodesic branched superminimal sphere of degree 3 and pull it back onto the Riemann surface via its branched double covering onto the sphere. In fact, that M is hyperelliptic is not fortuitous, since by looking into the interrelation between the Weierstrass points and the complete linear systems of degree 6 on nonhyperelliptic Riemann surfaces of genus 3 and 4, with the aid of Clifford's Theorem and the notion of correspondences between Riemann surfaces, we will assert in Theorem 2, Section 4, the following conclusion: For $g \geq 1$, a Riemann surface of genus g admits a nontotally geodesic branched superminimal immersion of degree 6 in S^4 if and only if the Riemann surface is hyperelliptic.

This naturally brings forward the question of classifying all nontotally geodesic superminiaml immersions of degree 6 for a given hyperelliptic Riemann surface. We have succeeded in carrying out the classification for $g \neq 2$ in Theorem 3, Section 4. Namely, all the nontotally geodesic branched superminimal immersions of degree 6 from a hyperelliptic Riemann surface of genus $g \geq 3$ into S^4 are just the pullback of nontotally geodesic branched superminimal spheres of degree 3 via the branched double covering. For $g = 1$,the closure of the space of nontotally geodesic branched superminimal tori of degree 6 in the moduli space is essentially a fiber bundle over the underlying torus, where each fiber in turn is a fiber bundle over a certain cubic curve whose fiber is a principal $PGL(2,\mathbb{C}) \times PGL(2,\mathbb{C})$-bundle over $G(2,4)$. This is to be proved in Section 6. It seems, suggested by $g = 1$, that a study on the Riemann Θ-function would lead to the classification when $g = 2$. In fact, our classification answers a question raised in [19] affirmatively for $d \leq 6$ as to whether the twisted cubic is the only curve in $\mathbb{C}P^3$ with a base-point-free complete g^3_d for $\mathcal{RAM}$ not to be injective.

For large degrees, exploring the "Weierstrass representation" mentioned earlier and the existence of nonspecial very ample line bundles for appropriate degrees, we prove in Theorem 4, Section 5, the existence of a nontotally geodesic branched superminimal immersion from any Riemann surface into S^4 as long as $d \geq 5g + 4$ for $g \geq 2$ (≥ 6 if $g = 1$). (This lower bound is sharp for $g = 1$.) Moreover, the dimension of each irreducible component of $\mathcal{M}_d(M)$ is bounded between $2d - 4g + 4$ and $2d - g + 4$. The upper bound is always achieved by the totally geodesic component, whereas the lower bound is realized by each nontotally geodesic component of $\mathcal{M}_6(T)$ for every torus T. Observe that when $g = 0$, the two dimension bounds are both equal to $2d + 4$. It is tempting to conjecture that the nontotally geodesic part of $\mathcal{M}_d(M)$ is of pure dimension $2d - 4g + 4$ for any Riemann surface (or at least for a generic) M of genus g. This would be true if the intersection of $Ker(\mathcal{R})$, the kernel of $\mathcal{R} : \mathbb{P}(\bigwedge^2(H^0(L)) \longrightarrow \mathbb{P}(H^0(K \otimes L^2))$, and the projective variety $\mathcal{Z} = \mathbb{P}(\{\omega \in \bigwedge^2(H^0(L)) : \omega \wedge \omega \wedge \omega = 0\})$ were transversal.

A study of the intersection of $Ker(\mathcal{R})$ and $\mathcal{Z}$ in the case $d = 6$ and $g = 1$ in Section 6 shows that the nontotally geodesic part of the moduli space $\mathcal{M}_6(T)$, where T is a torus, may be reducible (e.g., when T is the torus where the conformal structure is given by the Weierstrass constants $g_2 = 0$, $g_3 = 1$), so that the moduli space of branched superminimal tori with these conformal structures consist of more than three components (seven to be precise), although for a generic torus it is irreducible. It is likely that for a generic Riemann surface M of genus g, the nontotally geodesic part of $\mathcal{M}_d(M)$ is irreducible. Again, this would follow if the intersection of $Ker(\mathcal{R})$ and $\mathcal{Z}$ were transversal by a result in [7].

1. Twistor theory and superminimal immersions. Since Bryant's initial work ([2]) there have been many general investigations of minimal immersions in terms of the twistorial scheme, which we will briefly present in this section; for a detailed discussion and related references see [5], [6], [9]. Given an oriented Riemannian 4-manifold N, let $O(N)$ be the orthonormal frame bundle of N. Consider the bundle of pointwise orthogonal complex structures $O(N) \times_{O(4)} O(4)/U(2)$, which has two connected components Z_+ and Z_-, called the twistor spaces of N, consisting of those pointwise complex structures that are orientation-preserving and orientation-reversing, respectively. $Z_\pm$ is a 2-sphere bundle over N associated with $SO(4)$. The Levi-Civita connection on N then induces a connection on $Z_\pm$ which splits the tangent spaces of $Z_\pm$ into vertical and horizontal spaces, $TZ_\pm = V_\pm \oplus H_\pm$. Hence $TZ_\pm$ inherits naturally a Riemannian metric $<,>$ that coincides with that of N on $H_\pm$ and that of S^2 on $V_\pm$ such that $V_\pm$ is perpendicular to $H_\pm$. One can define a Hermitian structure J on Z by setting, at $u \in Z_\pm$, J to be the natural complex structure on V_u (the fiber of $Z_\pm$ is S^2 indentified with $\mathbb{C}P^1$), and to be u acting on H_u (u itself is a pointwise complex structure). The pair (Z_-, J) ((Z_+, J), respectively) turns out to be a complex manifold if and only if N is self-dual (anti-self-dual, respectively). Moreover, $(Z_-, J, <,>)$ ($(Z_+, J, <,>)$, respectively) is Kaehler-Einstein if N is Einstein with positive scalar curvature; in fact, Z_- (Z_+, respectively) is either $\mathbb{C}P^3$ or $F(1,2)$, where $N = S^4$ or $\mathbb{C}P^2$ with the standard metric, respectively.

Let $f : M \longrightarrow N$ be an immersion with the induced metric from a compact Riemann surface M into N. For each point p in M, if one assigns to f_*T_pM the natural orientation μ_p induced from M, then $(f_*T_pM)^\perp$ inherits a unique orientation τ_p such that $\mu_p \oplus \tau_p$ is the orientation of N at $f(p)$. Regarding μ_p and τ_p ($-\tau_p$, respectively) as complex structures on f_*T_pM

and $(f_*T_pM)^\perp$, one can define a map $\tilde{f_+} : p \longmapsto \mu_p \oplus \tau_p$ ($\tilde{f_-} : p \longmapsto \mu_p \oplus -\tau_p$, respectively) from M into Z_+ (Z_-, respectively), called the twistor lifts.

Now let e_1, e_2, e_3, e_4 be an adapted orthonormal frame of M so that (e_1, e_2) is a positively oriented frame on M, and let $\theta^a, \omega^a_b, 1 \leq a, b \leq 4$, be the coframe and the connection forms of N with respect to the adapted frame. Then $\omega^\alpha_i = \sum_j h^\alpha_{ij}\theta^j, 1 \leq i, j \leq 2, 3 \leq \alpha \leq 4$, where $\sum_{i,j} h^\alpha_{ij}\theta^i \otimes \theta^j$ is the second fundamental form. Set $H^\alpha = (h^\alpha_{11} + h^\alpha_{22})/2$, and $L^\alpha = (h^\alpha_{11} - h^\alpha_{22})/2 - \sqrt{-1}\, h^\alpha_{12}$. Consider the (1,0)-form $\phi = \theta^1 + \sqrt{-1}\,\theta^2$. One observes that $Q_f = (\sum_\alpha (L^\alpha)^2)\phi^4$ is a globally defined quartic form on M. Write $Q_f = S_+S_-\phi^4$, where $S_\pm = L^3 \mp \sqrt{-1}\, L^4$; $|S_\pm|$ are globally defined smooth functions. We say that f is an *isotropic* isometric immersion if $Q_f \equiv 0$, and f is *isotropic with positive spin (negative spin, respectively)* if $|S_+| \equiv 0$ ($|S_-| \equiv 0$, respectively).

The important fact is that the twistorial lift $\tilde{f_+}$ ($\tilde{f_-}$, respectively) is J-holomorphic if and only if f is isotropic with positive spin (negative spin, respectively). Furthermore, $\tilde{f_+}$ ($\tilde{f_-}$, respectively) is horizontal with respect to the spliting $TZ_+ = V_+ \oplus H_+$ ($TZ_- = V_- \oplus H_-$, respectively) if and only if f is minimal and $\tilde{f_+}$ ($\tilde{f_-}$, respectively) is J-holomorphic; f is said to be a superminimal immersion with positive spin (negative spin, respectively) in this case. It should be remarked that f is superminimal with both positive and negative spin if and only if f is totally geodesic; moreover, it is clear that reversing the orientation of N interchanges Z_+ and Z_-. It is for this reason that we consider only f with negative spin from now on.

2. Branched superminimal immersions in S^4. When we specialize N to S^4, the above formulation can be made explicit. To be more precise, one regards S^4 as $\mathbb{H}P^1$. Let τ be the universal quaternionic line bundle over S^4 with quaternionic multiplication on the right. Then one can identify TS^4 as $Hom_{\mathbb{H}}(\tau, \tau^\perp)$ where $\tau \oplus \tau^\perp = \mathbb{H}P^1 \times (\mathbb{H} \oplus \mathbb{H})$. Each v in τ_p, where p is the base point of v, can be regarded as an element $\tilde{v}$ in $Hom(T_pS^4, (\tau_p)^\perp_{\mathbb{R}})$ given by $\tilde{v}(f) = f(v)$ for f in $Hom_{\mathbb{H}}(\tau_p, (\tau_p)^\perp)$; $\tilde{v}$ is a real vector space isomorphism between T_pS^4 and $(\tau_p)^\perp_{\mathbb{R}}$ if $v \neq 0$. Since $(\tau_p)^\perp_{\mathbb{C}} = \mathbb{C} \oplus \mathbb{C}$ (complex multiplication on the right), it is clear that $\tilde{v}$ then induces a complex structure on T_pS^4, which is orientation-reversing. Now since the complex structure is unaltered by changing $\tilde{v}$ to $\tilde{v}\lambda$ for any $\lambda \in \mathbb{C}$, it follows that Z_- is $\mathbb{P}(\tau_{\mathbb{C}})$, which is $\mathbb{C}P^3$ with the Fubini-Study metric.

The horizontal distribution of $\mathbb{C}P^3 = Z_-$ is easy to describe: $T\mathbb{C}P^3 = V \oplus H$, where H is the kernel of a contact form whose pullback to $\mathbb{C}^4 \setminus \{0\}$

is

$$(z_0dz_1 - z_1dz_0 + z_2dz_3 - z_3dz_2)/\|z\|^2,$$

where $z_0, \cdots, z_3$ are the homogeneous coordinates of $\mathbb{C}P^3$. Hence a branched superminimal immersion of genus g and degree d in S^4 is the projection of a holomorphic curve $F : M \longrightarrow \mathbb{C}P^3$ of degree d and genus g satisfying (1).

We have seen in Section 0 that $\mathcal{RAM}(f_1) = \mathcal{RAM}(f_2)$ for a horizontal curve. Conversely, let L be a holomorphic line bundle of degree d over M, and let $z_0, \cdots, z_3$ be four sections without commom zeros in $H^0(L)$. If $\mathcal{RAM}\,(f_1) = \mathcal{RAM}\,(f_2)$, then there is a constant c^2 such that $[[z_0, z_1]] = -c^2[[z_2, z_3]]$ in view of (3); we may assume $c = 1$ without loss of generality. It follows that $[z_0 : z_1 : \pm z_2 : \pm z_3]$ will define two holomorphic maps $F_\pm$ of degree d from M to $\mathbb{C}P^3$ which satisfy (1). Therefore F can be reconstructed from the pair (f_1, f_2) up to the contact involution

$$\sigma : [z_0 : z_1 : z_2 : z_3] \longmapsto [z_0 : z_1 : -z_2 : -z_3]. \tag{5}$$

We choose to identify $[1 : 0]$ and $[0 : 1]$ in $\mathbb{H}P^1$ with the south and north poles of S^4, respectively. Then σ induces the geodesic symmetry about the south pole. This reduces to the construction in Loo ([12]) when the genus is zero.

In view of (4), define

$$\mathcal{N}_d^*(M) = \left\{ (f_1, f_2) \in R_d^1 \times R_d^1 : \begin{array}{c} \mu(f_1) = \mu(f_2), \\ \mathcal{RAM}(f_1) = \mathcal{RAM}(f_2), \\ \pi_1(f_1) \text{ and } \pi_1(f_2) \\ \text{have disjoint base loci} \end{array} \right\}.$$

Observe that $\mathcal{N}_d^*(M)$ recovers all the horizontal curves of degree d in $\mathbb{C}P^3$ except the ones where the pair (f_1, f_2) is a constant (if f_1 is constant then f_2 is constant by (1)), in which case the corresponding horizontal curves are of the form $[s : as : t : bt]$, where a, b are complex numbers and s, t are two sections of L_F; the projection of these horizontal curves into S^4 gives totally geodesic 2-spheres passing through both the north and south poles. To include these horizontal curves, we must enlarge $\mathcal{N}_d^*(M)$.

Recall that R_d^1 is a principal $PGL(2, \mathbb{C})$-bundle over G_d^1. As in [13], if we identify $\mathbb{C}P^3$ with the projectivization of the space of nonzero 2×2 matrices, there is a natural $PGL(2, \mathbb{C})$-action on $\mathbb{C}P^3$. Consider the associated bundle

$$\overline{R}_d^1 =: R_d^1 \times_{PGL(2,\mathbb{C})} \mathbb{C}P^3$$

over G^1_d. Fix $(s_1, s_2) \in R^1_d$ generated by a g^1_d spanned by s_1 and s_2. Any other (s'_1, s'_2) with $s'_i = \sum_j a_{ji} s_j$ is identified with $[(s_1, s_2), (a_{ji})] \in \overline{R}^1_d$. As a consequence, a horizontal curve of the form $[s : as : t : bt] \in \mathbb{C}P^3$ projects to

$$[(s,t), \begin{pmatrix} 1a \\ 00 \end{pmatrix}] \times [(s,t), \begin{pmatrix} 00 \\ 1b \end{pmatrix}]$$

in $\overline{R}^1_d \times \overline{R}^1_d$. Accordingly, we set

$$\mathcal{N}_d(M) = \left\{ (f_1, f_2) \in \overline{R}^1_d \times \overline{R}^1_d : \begin{array}{c} \overline{\mu}(f_1) = \overline{\mu}(f_2), \\ \mathcal{RAM}(f_1) = \mathcal{RAM}(f_2), \\ \pi_2(f_1) \textit{ and } \pi_2(f_2) \\ \text{have disjoint base loci} \end{array} \right\}, \qquad (6)$$

where

$$\pi_2 : \overline{R}^1_d \longrightarrow G^1_d$$

is the natural projection and

$$\overline{\mu} = \pi \circ \pi_2$$

with π given in (4); it is understood that $\mathcal{RAM}(f_1)$, for instance, is the ramification divisor of $\pi_2(f_1) \in G^1_d$. Then $\mathcal{N}_d(M)$ recovers $\mathcal{M}_d(M)$ up to the involution (5). In other word, $\mathcal{N}_d(M) = \mathcal{M}_d(M)/\sigma$.

Proposition 1 *Let $p : \mathcal{M}_d(M) \longrightarrow \mathcal{N}_d(M)$ be the covering map, and let $V_1, \cdots, V_k$ be the irreducible components of $\mathcal{N}_d(M)$. Then $p^{-1}(V_1), \cdots, p^{-1}(V_k)$ are the irreducible components of $\mathcal{M}_d(M)$. Furthermore, $\mathcal{M}_d(M)$ is connected if and only if $\mathcal{N}_d(M)$ is connected.*

Proof. We claim first that σ in (5) is homotopic to the identity map. Indeed, each orthogonal transformation on S^4 induces naturally an automorphism on $\mathbb{C}P^3$. Consider the geodesic symmetry about the south pole on S^4, which is an orientation-preserving isometry and hence is homotopic to the identity. This homotopy induces a homotopy on $\mathbb{C}P^3$ from the identity map to σ on $\mathbb{C}P^3$, and thus on $\mathcal{M}_d(M)$, which interchanges the two elements in each fiber of the map p. The first statement follows from the fact that σ, being homotopic to the identity map, must leave invariant each irreducible component of $\mathcal{M}_d(M)$. The second statement is a consequence of the first. □

3. Connectedness. From now on we consider $\mathcal{N}_d(M)$ in view of Proposition 1. To understand the space $\mathcal{N}_d(M)$, we will slice it by $L \in W_d^1$. Namely, fixing L we consider the space

$$\mathcal{N}_{d,L}(M) = \{(f_1, f_2) \in \mathcal{N}_d(M) : \pi\pi_2(f_1) = \pi\pi_2(f_2) = L\}$$

Clearly, $\mathcal{N}_d(M) = \bigcup_{L \in W_d^1} \mathcal{N}_{d,L}(M)$.

Let L be a line bundle of degree d, and let $t_1, t_2, \cdots, t_m$ be a basis of $H^0(L)$. The map

$$\mathcal{R} : s \wedge t \longmapsto [[s,t]], \tag{7}$$

where $[[s,t]]$ is given in (2), extends to a linear map from $\bigwedge^2 H^0(L)$ to $H^0(K \otimes L^2)$, which can be projectivized as a rational map, still denoted by $\mathcal{R}$, from $\mathbb{P}(\bigwedge^2(H^0(L)) \simeq \mathbb{C}P^{m(m-1)/2-1}$ to $\mathbb{P}(H^0(K \otimes L^2)) \simeq \mathbb{C}P^{2d+g-2}$. Note that $\mathcal{R}$ is completely determined by $\mathcal{R}(t_i \wedge t_j) = [[t_i, t_j]]$. Let $G(2, H^0(L))$ be the Grassmann manifold of 2-planes in $H^0(L)$; $G(2, H^0(L)) \hookrightarrow \mathbb{P}(\bigwedge^2(H^0(L)))$ via the plücker embedding $[[s,t]] \longmapsto s \wedge t$, where s and t span the plane (s,t).

Lemma 1 *The rational map $\mathcal{R}$ in (7) is regular on $G(2, H^0(L))$.*

Proof. If $\mathcal{R}(s \wedge t) = s \wedge dt - t \wedge ds = 0$, then $d(t/s) = 0$ so that $s \wedge t = 0$, which is impossible. □

Lemma 2 *$\mathcal{R}$ restricted to $G(2, H^0(L))$ is a finite map.*

Proof. $\mathcal{R}$ is induced by a linear map, and can therefore be regarded as a projection whose center does not intersect $G(2, H^0(L))$ by Lemma 1. Hence $\mathcal{R}$ is a finite map on $G(2, H^0(L))$ ([16]). □

Lemma 3 *If $(x,y) \in \mathcal{N}_{d,L}(M), x \neq y$, is represented by $\pi_2(x) = [e_1 \wedge e_2]$ and $\pi_2(y) = [e_3 \wedge e_4]$ in $G(2, H^0(L))$, then $e_1, \cdots, e_4$ are linearly independent in $H^0(L)$. Here "[ ]" denotes projectivization.*

Proof. Since $\mathcal{R}(\pi_2(x)) = \mathcal{R}(\pi_2(y))$, we have $\mathcal{R}(e_1 \wedge e_2 - \lambda e_3 \wedge e_4) = 0$ for some $\lambda \in \mathbb{C}$ on the Euclidean level; we may assume $\lambda = 1$ by rescaling. So $[v] = [e_1 \wedge e_2 - e_3 \wedge e_4]$ does not lie in $G(2, H^0(L))$ by Lemma 1. Hence $v \wedge v \neq 0$. This implies $e_1, \cdots, e_4$ are independent. □

In light of Lemma 3, we now restrict our consideration from $H^0(L)$ to a 4-dimensional linear subsystem $V_4 \subset H^0(L)$. Let $G(2, V_4) \subset G(2, H^0(L))$ be the Grassmann manifold of 2-planes in V_4.

Lemma 4 *$\mathcal{R}$ restricted to $G(2, V_4)$ is either one-to-one or a branched double covering onto its image.*

Proof. It is wellknown that $G(2, V_4) \subset \mathbb{P}(\bigwedge^2 V_4) \simeq \mathbb{C}P^5$ is a smooth hyperquadric . If $\mathcal{R}$ is not one-to-one, $\mathcal{R}$ restricted to $\mathbb{P}(\bigwedge^2 V_4)$ must have a center, which cannot intersect $G(2, V_4)$ by Lemma 1, and therefore must be a single point. This shows that $\mathcal{R}(\mathbb{P}(\bigwedge^2 V_4)) \simeq \mathbb{C}P^4$. Since $\dim G(2, V_4) = 4$ and $\mathcal{R}$ is a finite map, the image of $G(2, V_4)$ must be of dimension 4 and is therefore the entire $\mathbb{C}P^4$. The fact that $G(2, V_4)$ is a quadric implies that $\mathcal{R}$ is a branched double covering. □

The connectedness of $\mathcal{N}_d(M)$ is now immediate from Lemma 4 since one can always deform $(x, y) \in \mathcal{N}_d(M)$ to some (t, t) on the singular locus of $G(2, V_4)$ of $\mathcal{R}$.

The connectedness of $\mathcal{N}_d(M)$ has been recently proved for all S^n in [8].

4. Moduli space of small degree. From now on we denote $\dim H^0(L)$ by $h^0(L)$. One consequence of Lemma 3 is that to construct branched superminimal immersions which are not of the form $(f, Af) \in \mathcal{N}_d^*(M)$, where f is of degree d and $A \in PGL(2, \mathbb{C})$, i.e., which are not totally geodesic, it is necessary that one start with a line bundle $L \in W_d^1$ such that $h^0(L) \geq 4$, i.e., $L \in W_d^3$. However, there is no W_d^r when M is generic and the Brill-Noether number $(r + 1)(d - r) - rg < 0$. So we have the following.

Proposition 2 *Let M be a generic Riemann surface of genus $g \geq 1$. $\mathcal{N}_d(M)$ is empty if $d < (g+2)/2$. $\mathcal{N}_d^*(M)$ consists of (f, Af), where f is of degree d and $A \in PGL(2, \mathbb{C})$, if $(g+2)/2 \leq d < (3g+12)/4$; so all the corresponding branched superminimal immersions in S^4 are totally geodesic. Here, any Riemann surface of $g = 1, 2, 3$ is considered generic.*

Proof. Take $r = 1$ in the Brill-Noether number, which is < 0 if $d < (g+2)/2$, in which case there are no W_d^1 for a generic Riemann surface. Similarly, take $r = 3$ in the Brill-Noether number, which is < 0 when $d < (3g+12)/4$; hence there are no W_d^3 for a generic Riemann surface. Finally, for any Riemann surface of $g = 1, 2, 3$ with a line bundle L of a given degree within the bounds, one checks by the Riemann-Roch Theorem that $h^0(L) \leq 3$. □

On the other hand Clifford's Theorem enables us to look into the case of a small degree d for any Riemann surface. Recall first that Clifford's Theorem ([10]) states that if $L \in W_d^r \setminus W_d^{r+1}$ with $d \leq 2g - 2$, then $d \geq 2r$; furthermore if $d = 2r$, then either L is trivial, or $L = K$, the canonical bundle, or the Riemann surface M is hyperelliptic with the branched double covering $\phi : M \longrightarrow \mathbb{C}P^1$ and $L = (\phi^*\mathcal{O}(1))^r$.

Proposition 3 *$\mathcal{N}_d^*(M) = \{(f, Af) : f$ is of degree d and $A \in PGL(2, \mathbb{C})\}$ if $\min(g, 6) \geq d$. Hence the branched superminimal immersions from M into S^4 are all totally geodesic, except in the case when M is hyperelliptic and $d = 6$, in which case $\mathcal{M}_6(M)$ is isomorphic to $\mathcal{M}_3(\mathbb{C}P^1)$, the moduli space of horizontal rational curves of degree 3.*

Proof. If there is an $(x_0, y_0) \in \mathcal{N}_{d,L}(M)$ with $x_0 \neq y_0$, then $L \in W_d^r$ with $r \geq 3$ as mentioned earlier. By Clifford's Theorem, $6 \geq d \geq 2r \geq 6$. However this is possible only when $d = 6 = 2r$. Now $L \neq K$ since $d \neq 2g - 2$. Hence Clifford's Theorem infers that the Riemann surface is hyperelliptic, $L = (\phi^*\mathcal{O}(1))^3$ and $H^0(L)$ is generated by $(z)^i \circ \phi, 0 \leq i \leq 3$, where $\phi : M \longrightarrow \mathbb{C}P^1$ is the double covering and $z \in \mathbb{C}$ (one regards $\mathbb{C}P^1$ as $\mathbb{C} \cup \{\infty\}$). Therefore $G(2, H^0(L))$ comprises of $f \circ \phi$, where $f : M \longrightarrow \mathbb{C}P^1$ and $deg(f) \leq 3$. Now since $d(f \circ \phi) = df \circ d\phi$, we see that

$$\mathcal{RAM}(f \circ \phi) = \mathcal{RAM}(\phi) + \phi^{-1}(\mathcal{RAM}(f)).$$

It follows that $\mathcal{RAM}(f \circ \phi) = \mathcal{RAM}(g \circ \phi)$ if and only if $\mathcal{RAM}(f) = \mathcal{RAM}(g)$. Consequently the proposition will be true if we can verify that all the maps from M to $\mathbb{C}P^1$ of degree 6 come from $G(2, H^0(L))$. But this is the case since all the complete g_d^r with $d \leq g$ (in our case $d = 6$ and $r \leq 3$) on a hyperelliptic curve is of the form $rg_2^1 + p_1 + p_2 + \cdots + p_{d-2r}$, where no two of the points p_i are invariant under the involution of M induced by ϕ and g_2^1 is the linear system corresponding to ϕ ([10]); hence the $g_d^r, r \leq 2$, will be ruled out since they have base locus $p_1, \cdots, p_{d-2r}$. □

Remark 1 *In the same vein as in the proof of Proposition 3, let M be a hyperelliptic curve and $d \leq g$. Then the moduli space of branched superminimal immersions from M into S^4 is isomorphic to that of branched superminimal spheres of degree $d/2$.*

Before proceeding with further examples of small degree, we first consider a general situation. Let $t_1, \cdots, t_m$ span $H^0(L)$. Consider the curve $\psi : M \longrightarrow \mathbb{C}P^{m-1}$ given by $\psi(p) = [t_1(p); \cdots, t_m(p)]$. The first associated curve ψ_1 of ψ, i.e., the set of the tangents of ψ, lies in $G(2, m)$ indentified with $G(2, H^0(L))$. Via the plücker embedding, $\psi_1 \in \mathbb{P}(\bigwedge^2(H^0(L))$.

Lemma 5 *Let k be the dimension of the smallest linear subspace containing $\psi_1 \in \mathbb{P}(\bigwedge^2(H^0(L))$. Then the dimension of the center of the projection $\mathcal{R}$ in (7) is equal to $m(m-1)/2 - k - 2$.*

Proof. Observe first that in homogeneous coordinates $\psi_1 = [\psi \wedge \psi^{'}] = [\cdots : [[t_i, t_j]] : \cdots]$ (see (2) for notation), where we use ψ and $\psi^{'}$ to also denote the Euclidean lift and derivative of ψ. Hence any linear relation $\sum a_{ij}[[t_i, t_j]] = 0$ gives rise to the element $\sum a_{ij} t_i \wedge t_j$, which lies in the center of $\mathcal{R}$ in view of (7), and vice versa. □

Lemma 6 *Let* $[1 : z^{1+\alpha_1} : z^{2+\alpha_1+\alpha_2} : z^{3+\alpha_1+\alpha_2+\alpha_3}]$ *be the canonical form of a linearly full curve* $\psi \in \mathbb{C}P^3$ *around* $z = 0$*. Here we only display the first term in each Taylor series. Then* ψ_1*, the first associated curve of* ψ*, is linearly full in* $\mathbb{C}P^5 \supset G(2,4)$ *if* $\alpha_1 \neq \alpha_3$*.*

Proof. Assume $\alpha_3 < \alpha_1$. A straightforward computation shows that ψ_1 assumes the form $[1 : z^a : z^b : z^c : z^d : z^e]$, where $a = 1 + \alpha_2, b = 2 + \alpha_2 + \alpha_3, c = 2 + \alpha_1 + \alpha_2, d = 3 + \alpha_1 + \alpha_2 + \alpha_3$ and $e = 4 + \alpha_1 + 2\alpha_2 + \alpha_3$. It follows that $a < b < c < d < e$. So the curve is linearly full in $\mathbb{C}P^5$. □

4.1. The case when $d \leq 5$. We first study the crucial case when $g = 2$ and $d = 5$ so that $d = 2g + 1$. Let L be a line bundle of degree 5 over M of genus 2; $h^0(L) = 4$ by the Riemann-Roch Theorem. As mentioned before Lemma 5, any basis $t_1, \cdots, t_4 \in H^0(L)$ generates a curve $\psi : M \longrightarrow \mathbb{C}P^3$ of degree 5 which is an embedding in our case (any L of degree $d \geq 2g + 1$ is very ample). Conversely, a plane cut of any embedded space curve of $g = 2$ and $d = 5$ in $\mathbb{C}P^3$ gives a line bundle of degree 5. From now on we identify M with $C =: \psi(M) \in \mathbb{C}P^3$. Pick a point p on C and consider the projection π_p in $\mathbb{C}P^3$ whose center is p. $C' =: \pi_p(C)$ is a curve of degree 2 or 4 in $\mathbb{C}P^2$ because π_P has mapping degree 4. If $deg(C') = 2$, then C' is a conic. π_p may then be regarded as a branched double covering from C onto the Riemann sphere; hence the canonical bundle $K = p_1 + p_2$, where $\{p_1, p_2\} = \pi_p^{-1}(x)$ for any $x \in C'$. (For simplicity in notation, we regard "=" in $K = p_1 + p_2$, etc, as the divisor $p_1 + p_2$ defining K.) Now pick a line joining x and some y on C' with $\pi_p^{-1}(x)$ as given above and $\pi_p^{-1}(y) = \{p_3, p_4\}$. Then $D = p + p_1 + p_2 + p_3 + p_4$ is a plane cut of C defining L. It follows that

$$L = p + 2K.$$

On the other hand if $deg(C') = 4$, then C' has a unique double point x by the genus formula ([15]). Let $\{q_1, q_2\} = \pi^{-1}(x)$. Then p, q_1, q_2 are collinear. The projection whose center is the line $\overline{pq_1q_2}$ is a meromorphic function of order 2 whose poles are, say, p_1 and p_2 so that $K = p_1 + p_2$. Now $D = p + q_1 + q_2 + p_1 + p_2$ is a plane cut of C defining L. Hence

$$L = p + q_1 + q_2 + K,$$

where $q_1 + q_2 \neq K$.

Proposition 4 *Let M be a Riemann surface of genus 2. Then all the branched superminimal immersions of degree 5 from M into S^4 are totally geodesic.*

Proof. For $0 \leq i \leq 3$ and an arbitrary point q, we have $h^0(L - iq) = 4 - i + h^0(K - L + iq)$ by the Riemann-Roch Theorem, which is $4 - i$ if $i \leq 2$ since the degree of $K - D + iq$ is then negative. Now let $i = 3$.

Case(1). $L = p + 2K$. Then $H^0(K - L + 3q) = H^0(3q - p - 2s)$ with $K = 2s$ for some fixed Weierstrass point s chosen once and for all (recall that M is hyperelliptic).

If $h^0(3q - p - 2s) \neq 0$, there will be a meromorphic function f assuming the only pole of order at most 3 at q and zeros of order at least 1 and 2 at p and s, respectively. f cannot be of order 3; for otherwise, q and $2s$ are the only zeros of f, so that if we let ω be a holomorphic form whose zero is $2s$, then $f^{-1}\omega$ will be a meromorphic form with a single pole q of order 1, which is absurd. Thus f can only be of order 2. However, this implies that q will eliminate either p or s.

If $q = p$, then $2p = 2s$ and so $L = 5p$ with p a Weierstrass point. Now $h^0(ip) = 1, 1, 2, 2, 3, 4$ for $i = 0, 1, 2, 3, 4, 5$, respectively, since p is a Weierstrass point. We have $h^0(L - ip) = h^0((5 - i)p) = 4, 3, 2, 2, 1, 1, 0$ for $i = 0, 1, 2, 3, 4, 5, 6$, respectively. It follows that near p, the curve ψ assumes the parametric form $[1 : z : z^3 : z^5]$, so that $(\alpha_1, \alpha_2, \alpha_3)$ given in Lemma 6 is $(0, 1, 1)$; in particular $\alpha_1 \neq \alpha_3$ at p. Lemma 6 then implies that the first associated curve of ψ is nondegenerate in $\mathbb{C}P^5$. Lemma 5 in turn asserts that $\mathcal{R}$ has no center, i.e., $\mathcal{R}$ is injective. In other words, the branched superminimal immersion constructed is totally geodesic, which is what we intend to conclude.

If $q \neq p$ and so $q = s$. But then f will be a meromorphic function of order 1 with pole s and zero p, which is impossible unless $p = s = q$, so that one more time we obtain $L = 5p$ with p a Weierstrass point.

Therefore we may now assume that $h^0(3q - p - 2s) = 0$, i.e., $h^0(L - 3q) = 1$ for all q. In summary, we have $h^0(L - iq) = 4 - i, 0 \leq i \leq 3$, for all q. This is equivalent to saying that near any q, the curve ψ is of the form $[1 : z : z^2 : z^m]$ with $m \geq 3$; in particular, $\alpha_1 = \alpha_2 = 0$ for all points. However, there must be a point at which $\alpha_3 \neq 0$ by the Plücker formula ([10]), and thus at this point $\alpha_1 \neq \alpha_3$. Hence again $\mathcal{R}$ is injective and the branched superminimal immersion is totally geodesic.

Case(2). $L = p + q_1 + q_2 + K$ with $q_1 + q_2 \neq K$. As explained above, C' must be a curve of degree 4 in $\mathbb{C}P^2$ having only an ordinary double point. Since ψ is embedded, at a ramified point p the curve ψ is of the form $[1 : z : z^{2+\alpha_2} : z^{3+\alpha_2+\alpha_3}]$, where $\alpha_1 = 0$. If $\alpha_3 \neq 0$, then $\alpha_1 \neq \alpha_3$ and we are done by Lemmas 5 and 6. Hence we may assume $\alpha_1 = \alpha_3 = 0$ at all ramified points p. We claim that this case cannot occur. To this end, observe that the projection π_p in $\mathbb{C}P^3$ with center p maps C to C' whose only singularity, being the image of p, is a cusp of the form $(z^{1+\alpha_2}, z^{2+\alpha_2})$ in affine coordinates, so that $\alpha_2 = 1$ since the singularity must be an ordinary simple cusp. Thus $(\alpha_1, \alpha_2, \alpha_3) = (0, 1, 0)$ at all ramified points p. Now the Plücker formula $\sum_{k=1}^{3}(4 - i)\alpha_k = 32$ ([10]) implies that there are 16 ramified points for ψ. On the other hand, since the tangent line to ψ at a ramified point is of contact order 3, we must have $p = q_1 = q_2$ in $L = p+q_1+q_2+K$. Hence $L = 3p+K$ with $2p \neq K$ for all ramified points p; in particular p is not a Weierstrass point. Fixing one ramified point p_0, for any ramified point $p \neq p_0$ we have $L - K = 3p_0 = 3p$, so that there is a meromorphic function assuming the single pole and zero of order 3 at p_0 and p, respectively. However, $h^0(3p_0) = 2$, i.e., $3p_0$ defines a single g^1_3, we therefore see that all the ramified points belong to this g^1_3, each of ramification index 2. In particular, the total ramification index of this g^1_3 is ≥ 32, which is absurd, since the total ramification is 8 by the Riemann-Hurwitz formula. □

We are now ready to characterize all branched superminimal immersions of degree ≤ 5.

Theorem 1 *Let M be a Riemann surface of genus $g \geq 1$. Then all branched superminimal immersions of degree $d \leq 5$ from M into S^4 are totally geodesic.*

Proof. Proposition 9 below in Section 6 solves the case when $g = 1$. Proposition 2 takes care of $g = 2$ when $d \leq 4$ while $d = 5$ is handled by Proposition 4. The case $g = 3$ follows from Proposition 2. For $g = 4$, Proposition 3 gives the result as long as $d \leq 4$. However, when $g = 4$ and $d = 5$, we have $d < 2g - 2$; hence Clifford's Theorem implies that $d \geq 2r$, i.e., $h^0(L) \leq 3$ for any bundle L of degree 5. Finally, Proposition 3 settles $g \geq 5$. □

4.2. The case when $d = 6$. We first study the case $g = 3$ and $d = 6$ so that $d = 2g$. Let M be a hyperelliptic Riemann surface of genus 3 and let L be a line bundle over M of degree 6. As before let ψ be the curve in $\mathbb{C}P^3$ associated with L. Since $deg(L-p-q) = deg(K)$, we see that $L = K+p+q$ if and only if $h^0(L - p - q) = 3$ if and only if ψ is not embedded (recall that

ψ is embedded if and only if $h^0(L-p-q) = h^0(L)-2$ for all p and q ([10])). Notice that the curve ψ is not an immersion at p if $L = K + 2p$, whereas when $p \neq q$, ψ is an immersion but is not one-to-one when $L = K + p + q$.

Assume now that ψ is an embedded curve so that $L \neq K+x+y$ for any x and y. Identify M with $C = \psi(M)$. Pick a point $p \in C$. Let $C' = \pi_p(C)$ be the projection of C where π_p has center p. Since $deg(C') = 5$, C' has three ordinary double points x, y, z by the genus formula for a generic point p; in general these singularities may collapse so that higher order singularity may result. Let $\{x_1, x_2\}, \{y_1, y_2\}, \{z_1, z_2\}$ be the preimages of x, y, z, respectively, via π_p. The pair in each set is collinear with p; denote these three lines by l_1, l_2, l_3, respectively. The projections π_1, π_2, π_3 whose centers are l_1, l_2, l_3, respectively, are meromorphic functions of order 3 whose poles are, say, $\{p_1, p_2, p_3\}, \{q_1, q_2, q_3\}, \{r_1, r_2, r_3\}$, respectively, so that $h^0(p_1+p_2+p_3) = 2$ by nonhyperellipcy, or equivalently, $h^0(K-p_1-p_2-p_3) = 1$. In other words, there is a point p_0 such that p_0, p_1, p_2, p_3 are collinear on the canonical curve ϕ_K embedded in $\mathbb{C}P^2$ so that $K = p_0 + p_1 + p_2 + p_3$. Likewise there are q_0, r_0 collinear with $q_i, r_i, 1 \leq i \leq 3$, respectively, on ϕ_K. Since $D = p + x_1 + x_2 + p_1 + p_2 + p_3$ is a hyperplane cut defining L, we see that ([15])

$$L = p + x_1 + x_2 + K - p_0. \tag{8}$$

Similar identities hold when x_i are replaced by y_i and z_i and p_0 by q_0 and r_0, respectively, for $1 \leq i \leq 2$. In particular, $x_1 + x_2 - p_0 = y_1 + y_2 - q_0$ by (8), i.e., $x_1 + x_2 + q_0 = y_1 + y_2 + p_0$, or in other words x_1, x_2, q_0 are collinear on ϕ_K; similarly x_1, x_2, r_0 are collinear on ϕ_K. We see then that x_1, x_2, q_0, r_0 are collinear on ϕ_K so that $K = x_1 + x_2 + q_0 + r_0$. Substituting this into (8) gives

$$L = 2K + p - p_0 - q_0 - r_0. \tag{9}$$

Sublemma 1 *Notation is as above. Let ψ be immersed in $\mathbb{C}P^3$ and let p be a point at which the first associated curve of ψ is singular (i.e., $h^0(L-3p) = 3$). Then there is a meromorphic function of order 3 whose only pole is p. In particular, p is a Weierstrass point. Moreover,*

$$L = s_1 + s_2 + s_3 + 3p,$$

where s_1, s_2, s_3 are collinear on ϕ_K.

Proof. Assume first that ψ is embedded, we consider the correspondence $T(p) = p_0 + q_0 + r_0$. Now T is of valence -1, i.e., $T(p) - p$ is independent of p, which is the case since $T(p) - p = 2K - L$ by (9). Moreover, T

has no united points, i.e., there are no points p for which $p \in T(p)$, which follows because $p \in T(p)$ would force, say $p = p_0$, and thus by (8) we have $L = K + x_1 + x_2$ so that ψ would be singular. Therefore, the Cayley-Brill formula ([10]) asserts that $deg(T^{-1}) = 3$, i.e., for each point p_0, there are three points p, p', p'' such that $p_0 \in T(p), T(p'), T(p'')$. By the definition of p_0, this means that the points p_1, p_2, p_3 introduced above belong to three plane cuts through p, p', p'', respectively; in particular, p_1, p_2, p_3 are collinear. The projection whose center is the line $\overline{p_1p_2p_3}$ is a meromorphic function H of order 3, whose poles may be chosen to be p, x_1, x_2 since these six points are coplanar.

Since we assume that ψ is embedded, $h^0(L - 3p) = 3$ means that the tangent line to ψ at p has contact order at least 3 and so $p = x_1 = x_2$; hence the function H defined above is a function with a single pole of order 3 at p. By (8), we have $L = K + 3p - p_0 = p_1 + p_2 + p_3 + 3p$.

On the other hand, if ψ is immersed but not embedded, then we have $L = K + x + y$ with $x \neq y$. Now $h^0(L - 3p) = 3$ is equivalent to $h^0(3p - x - y) = 2$ by the Riemann-Roch Theorem, which gives the existence of such a function H of order 3 whose only pole is p. In particular, let s_1, x, y be the zeros of H. Then $3p = x + y + s_1$ and so there is a point s_2 such that $3p + s_2 = x + y + s_1 + s_2 = K$. Substituting this into $L = K + x + y$ yields $L = x + y + s_2 + 3p$ again. □

Sublemma 2 *Let p be an immersed point of a nondegenerate nonhyperelliptic curve ψ of degree 6 in $\mathbb{C}P^3$. Suppose the first associated curve of ψ is singular at p. Then the tangent line to ψ at p is of contact order 3. In particular, $\alpha_1 = 0$ and $\alpha_2 = 1$ at p.*

Proof. The contact order must be at least 3. If the contact order is 4, then the projection in $\mathbb{C}P^3$ whose center is the tangent line at p will be a meromorphic function of order at most 2, which is impossible since the curve is nonhyperelliptic. □

Proposition 5 *Let M be a nonhyperelliptic Riemann surface of genus 3. Then all the branched superminimal immersions from M into S^4 of degree 6 are totally geodesic.*

Proof. Suppose there is a nontotally geodesic branched superminimal immersion generated by a line bundle of degree 6. As usual let ψ be the curve in $\mathbb{C}P^3$ associated with L.

Case(1): ψ is immersed. By Lemma 6, $\alpha_1 = \alpha_3 = 0$ for all points. Take a ramified point q of ψ. Then $(\alpha_1, \alpha_2, \alpha_3) = (0.1.0)$ by Sublemma 2. Now the formula $\sum_{k=1}^{3}(4-k)\alpha_k = 48$ asserts that there are 24 ramified points on ψ, while Sublemma 1 says that these 24 points are all Weierstrass points. On the other hand, the Plücker formula applied to the canonical curve ϕ_K, which is embedded in $\mathbb{C}P^2$, gives $\sum(2\beta_1+\beta_2) = (g-1)g(g+1) = 24$ summed over all Weierstrass points, which were just proved to be ≥ 24 in number, where ϕ_K assumes the parametric form $[1 : z^{1+\beta_1} : z^{2+\beta_2}]$. Since $\beta_1 = 0$ for all p, we see that there are exactly 24 Weierstrass points with $\beta_2 = 1$ for all of them. In other words, all Weierstrass points are ordinary flexes.

Now given two Weierstrass points p and p', by Sublemma 1 we have $L = K + 3p - s_p = K + 3p' - s_{p'}$, for some points s_p and $s_{p'}$. Let $\tilde{p} + 3p = K = \tilde{p'} + 3p'$ (the tangent line to ϕ_K at p intersects ϕ_K at $\tilde{p}$). Then we see that $\tilde{p} + s_p = \tilde{p'} + s_{p'}$. Therefore, either $\tilde{p'} = \tilde{p}$, or $\tilde{p'} = s_p$ since M is nonhyperelliptic. Thus fixing p, the 24 Weierstrass points p' are divided into the set S_1 where $\tilde{p'} = \tilde{p}$, in which case the tangent lines to ϕ_K at p' are all through $\tilde{p}$, and the set S_2, where $\tilde{p'} = s_p$, in which case the tangent lines to ϕ_K at p' are all through s_p. We may thus assume that S_1 contains at least 12 Weierstrass points without loss of generality. However, the projection in $\mathbb{C}P^2$ whose center is p on ϕ_K gives a meromorphic function π_p of order 3 for which all p' in S_1 have ramification index 2 except when $p' = p$, where the ramification index is 1. Now the Riemann-Hurwitz formula says that the total ramification for π_p is 10, while the sub-total ramification over these Weierstrass points in S_1 is at least 23, which is a contradiction.

Case(2): ψ is not immersed. Then $L = K + 2p$ for some p. By the Riemann-Roch Theorem $h^0(L - ip) = 4 - i + h^0((i-2)p) = 4, 3, 3, 2, 1$ for $i = 0, 1, 2, 3, 4$, respectively, so that with the fact that $\alpha_1 = \alpha_3$ for all points we have, near p, that ψ is of the parametric form $[1 : z^2 : z^3 : z^5]$ with $(\alpha_1, \alpha_2, \alpha_3) = (1, 0, 1)$ at p. The existence of z^5 implies $h^0(L - 5p) = 1$ so that $h^0(3p) = 2$ by the Riemann-Roch Theorem; in particular, p is a Weierstrass point.

Let $q \neq p$ be a ramified point. $h^0(L - iq) = 4 - i + h^0(iq - 2p) = 4, 3, 2$ for $i = 0, 1, 2$, respectively. Hence q is an immersed point. Sublemma 2 then infers that $(\alpha_1, \alpha_2, \alpha_3) = (0, 1, 0)$ so that ψ is of the form $[1 : z : z^3 : z^4]$ near q. As a consequence of the nonexistence of z^2, we have $h^0(L - 3q) = 2$, or equivalently $h^0(3q - 2p) = 1$; in particular q is also a Weierstrass point and there is a point s_0 such that $3q = 2p + s_0$. Let s_1 and s_2 be such that $s_1 + 3q = K = s_2 + 3p$ (p and q are Weierstrass points which have contact of order at least 3 to ϕ_K). Then $p + s_2 = s_0 + s_1$. Hence either $3q = 3p$, in which case the tangent lines to ϕ_K at p and q pass through $s_1 = s_2$, or

$p = s_1$, in which case the tangent lines to ϕ_K at q pass through p; we divide such points q into two sets U_1 and U_2, respectively. We are now in a familiar situation that we saw in Case(1). The number of ramified points of ψ is 23 (total ramification at p is 4 and is 2 at $q \neq p$), so that we may assume U_1 contains at least 12 of them for instance. However, the projection with center p_2 in $\mathbb{C}P^2$ is a meromorphic function of order 3 which has $q \in U_1$ as ramified points and whose total ramification is 10, which is absurd. □

The upper limit of the degree d of a special line bundle L (L is special if $h^0(K \otimes L^{-1}) \neq 0$) is $2g-2$ by the Riemann-Roch Theorem. Let $g = 4$ and $d = 2g - 2 = 6$. Let M be a nonhyperelliptic Riemann surface of genus 4. Recall that M is the intersection of a quartic surface Q and a cubic surface C in $\mathbb{C}P^3$ ([10]). If Q is nonsingular, Q has two 1-parameter families of independent rulings L_1 and L_2, where the 1-parameter t for L_1 (s for L_2, respectively) run along a fixed line in L_2 (a fixed line in L_1, respectively), such that any two different lines in the same ruling are not coplar whereas any two lines from the two different rulings are coplanar. On the other hand, Q degenerates to a cone if it is singular and the two rulings coincide. Each line $l_t \in L_1$ ($l_s^{'} \in L_2$, respectively) intersects the cubic surface C in three points p_1, p_2, p_3. The two projecions from $\mathbb{C}P^3$ to $\mathbb{C}P^1$ whose centers are the lines l_t and $l_s^{'}$ give rise to two meromorphic functions of order 3 on M; hence there are at least two g_3^1. To see that there are exactly two g_3^1 for a nonsingular Q, let $q_1 + q_2 + q_3$ be a divisor defining a g_3^1. Since $h^0(q_1 + q_2 + q_3) = h^0(K - q_1 - q_2 - q_3)$ and in turn equals the number of independent planes containing $q_1, q_2, q_3 \in \mathbb{C}P^3$, it follows that $h^0(q_1 + q_2 + q_3) = 2$ and q_1, q_2, q_3 are collinear. The line through q_1, q_2, q_3 must belong to one of the rulings; therefore there are exactly two g_3^1 for a nonsingular Q. In particular, if Q degenerates to a cone, then $l_t = l_s^{'}$ and so there is a unique g_3^1.

Proposition 6 *Let M be a nonhyperelliptic Riemann surface of genus 4. Than all the branched superminimal immersions of degree 6 from M into S^4 are totally geodesic.*

Proof. Since $d = 2g - 2$, the line bundle L must be the canonical bundle so that the corresponding curve ψ is nothing but the canonical curve in $\mathbb{C}P^3$, which is embedded. We identify M with $C =: \psi(M)$. We recall that on a canonical curve, a point p is an unramified point for all of the associated curves of ψ if and only if p is a non-Weierstrass point. Accordingly, we assume that q is a Weierstrass point in what follows. By Lemma 6 and

Sublemma 2, once more we have $(\alpha_1, \alpha_2, \alpha_3) = (0, 1, 0)$ for all q. We claim that this case cannot occur. For, first note that the Plücker formula gives that the number of Weierstrass points is $(g-1)g(g+1)/2 = 30$. However, since 3 is not a Weierstrass gap value at q we see that all $3q$ belong to the two g_3^1 if the Quadric surface Q is nondegenerate; one of these g_3^1 therefore contains at least 15 Weierstrass points q at which the ramification index of this g_3^1 is 2. By the Riemann-Hurwitz formula, the total ramification of the g_3^1, which is 12, must be greater than or equal to the sub-total ramification index evaluated at these Weierstrass points, which is at least 30. This is a contradiction. □

Remark 2 *Equivalently put, Propositions 4 through 6 say that all nonhyperelliptic space curves of degree 5 and genus 2 and of degree 6 and genus 3 and 4 have nondegenerate first associated curves in* $\mathbb{C}P^5$.

We are ready to characterize the Riemann surfaces of genus ≥ 1 for which there exist nontotally geodesic branched superminimal immersions into S^4.

Theorem 2 *Let* M *be a Riemann surface of genus* $g \geq 1$. M *admits a nontotally geodesic branched superminimal immersion of degree 6 into* S^4 *if and only if* M *is hyperelliptic.*

Proof. If M is hyperelliptic of any genus, then M admits a nontotally geodesic branched superminimal immersion. More precisely, let $\phi : M \longrightarrow \mathbb{C}P^1$ be the branched double covering and let (f_1, f_2) be a pair of meromorphic functions on $\mathbb{C}P^1$ which gives rise to a nontotally geodesic branched superminimal sphere (of order 3). Then $(f_1 \circ \phi, f_2 \circ \phi)$ is a pair which generates a nontotally geodesic branched superminimal immersion on M (of order 6). Conversely, Proposition 3 takes care of $g \geq 6$. For $g = 4, 5$, we have $2g - 2 \geq 6 = d$. Hence $h^0(L) = 4$ by Clifford's Theorem if one can construct a nontotally geodesic branched superminimal immersion on L. Now L is not the canonical bundle for $g = 5$ since $2g - 2 \neq d$; Clifford's Theorem then concludes that M is hyperelliptic. Finally, Propositions 5 and 6 finish the cases $g = 3, 4$. □

4.3. Classification of $\mathcal{M}_6(M)$ when $g \geq 3$. First consider a hyperelliptic Riemann surface M of genus 3. Let L be a line bundle of degree 6 over M and let ψ be the curve of degree 6 associated with L in $\mathbb{C}P^3$. Assume ψ is embedded and identify M with $C =: \psi(M)$. For a point $p \in M$ consider the projection π_p whose center is p. $C' =: \pi_p(C)$ is a curve of degree 5 in $\mathbb{C}P^2$

which has a unique triple point as singularity by the genus formula and the fact that a hyperelliptic Riemann surface of genus ≥ 3 has no meromorphic functions of order 3 (so that the singularity cannot be a double point). Let this singular point be x and let $\pi^{-1}(x) = \{p_1, p_2, p_3\}$. As before, p, p_1, p_2, p_3 are collinear and the projection whose center is this line is a meromorphic function of order 2. Let the pole of this function be a Weierstrass point w chosen once and for all. We have

$$L = 2w + p + p_1 + p_2 + p_3. \tag{10}$$

Note that $K = 4w$. Consider the correspondence $T(p) = p_1 + p_2 + p_3$. T has valence 1 since $T(p) + p = L - 2w$. Furthermore $deg(T^{-1}) = 3$; for otherwise, if $p_1 \in T(q)$ for $q \neq p, p_2, p_3$, then $p, q, T(p), T(q)$ would be coplanar so that $deg(C) \geq 7$. It follows from the Cayley-Brill formula that T has 12 united points. Now since the tangent line to ψ at a ramified point p is of contact order ≥ 3, we see that we may assume $p_2 = p_3 = p$ in (10), so that on the one hand

$$L = 2w + 3p + p_1, \tag{11}$$

and on the other hand p is a united point. Hence the number of ramified points is ≤ 12.

Proposition 7 *Let M be a hyperelliptic Riemann surface of genus 3. A nontotally geodesic branched superminimal immersion of degree 6 from M into S^4 is the pullback of a branched superminimal sphere of degree 3 via the branched double covering $\phi : M \longrightarrow \mathbb{C}P^1$.*

Proof. Let L be a line bundle of degree 6 generating the nontotally geodesic branched superminimal immersion. As before let ψ be the holomorphic curve in $\mathbb{C}P^3$ associated with L.

In what follows, we will assume that $L \neq 6w$; otherwise it is just the conclusion of this proposition since $\psi = [1 : \phi : \phi^2 : \phi^3]$ then.

Case(1). ψ is nonsingular. $\alpha_1 = \alpha_3 = 0$ for all $q \in M$ by Lemma 6. Let p be a ramified point of ψ. By (11) $h^0(L - ip) = 4 - i + h^0(2w + (i-3)p - p_1) = 4, 3, 2, 2$ if $i = 0, 1, 2, 3$ since ψ is an embedding and its first associated curve is singular at p. Now $h^0(L - 4p) = h^0(2w + p - p_1)$ and moreover $h^0(2w + p) = h^0(2w)$ because M has no meromorphic functions of order 3. We see that $h^0(L - 4p) = h^0(2w + p - p_1) = 2$ or 1 if $p = p_1$ or $p \neq p_1$, respectively. (If $h^0(2w + p - p_1) = 2$ when $p \neq p_1$, then $p_1 = w$ and $w + p = 2w$ since $h^0(w + p) = 2$; hence $p = p_1 = w$ and $L = 6w$ by (11), which is excluded.) It follows that either $p = p_1$ where $(\alpha_1, \alpha_2, \alpha_3) = (0, 2, 0)$ and $L = 4p + 2w$, or $p \neq p_1$ and $(\alpha_1, \alpha_2, \alpha_3) = (0, 1, 0)$.

We now estimate the number of ramified points p for which $L = 4p + 2w$. Pick one such point p_0. Any other such p satisfies $4p = 4p_0 = L - 2w$. On the other hand, $h^0(4p_0) = 2$ by the Riemann-Roch Theorem since $K \neq 4p_0$ (or else $L = 6w$). We assert then that all these ramified points belong to the g^1_4 generated by $4p_0$, each of ramification index 3. Since the Riemann-Hurwitz formula says that the total ramification index of this g^1_4 is 12, it follows that there are at most 4 ramified points p such that $L = 4p + 2w$. On the other hand, the formula $\sum_{k=1}^{3}(4 - k)\alpha_k = 48$ for ψ implies that there are at least 16 ramified points such that $L = 3p + p_1 + 2w$ with $p \neq p_1$. This is a contradiction since we mentioned preceding this proposition that there are at most 12 ramified points.

Case(2). ψ is singular. $L = K + x + y$ for some x and y; since we assume that $K \neq 6w$, we have $x + y \neq 2w$. Now $h^0(L - iw) = 4 - i + h^0(iw - x - y) = 4, 3, 2$ for $i = 0, 1, 2$. If $h^0(L - 3w) = 1$, then since $h^0(L - 4w) = h^0(x + y) = 1$ (recall $x + y \neq 2w$) we have that near w, the curve ψ assumes the parametric form $[1 : z : z^2 : z^m]$ with $m \geq 4$ so that $\alpha_1 \neq \alpha_3$ at w, which is ruled out by Lemma 6. Thus $h^0(L - 3w) = 2$, i.e., $h^0(3w - x - y) = 1$. Hence there is a point z such that $3w = x + y + z$. However, this forces $x = w$ or $y = w$; for on the one hand one of x, y and z must equal w since there are no meromorphic function of order 3, and on the other hand $z \neq w$, or else $x + y = 2w$. We may thus assume without loss of generality that $x = w$, so that $L = 5w + y$ with $y \neq w$. Now $h^0(L - 5w) = h^0(y) = 1$, and $h^0(L - 6w) = h^0(y - w) = 0$. We conclude that near w, the curve ψ is of the form $[1 : z : z^3 : z^5]$, so that $\alpha_1 \neq \alpha_3$ at w, which is impossible by Lemma 6. □

We are now in a position to classify $\mathcal{M}_6(M)$ when $g \geq 3$.

Theorem 3 *Let M be a hyperelliptic surface of genus $g \geq 3$. Then $\mathcal{M}_6(M) = V_1 \cup V_2$, where V_1 is the totally geodesic part $\simeq \overline{R}^1_6$ (see Section 2 for notation), and V_2 is isomorphic to the nontotally geodesic part of $\mathcal{M}_3(\mathbb{C}P^1)$. V_1 and V_2 are identified along the singular locus of $\mathcal{M}_3(\mathbb{C}P^1)$. In particular, nontotally geodesic branched superminimal immersions of degree 6 from M into S^4 are the pullback of nontotally geodesic branched superminimal spheres of degree 3 via the branched double covering of M onto $\mathbb{C}P^1$. Furthermore, $\mathcal{M}_6(M) \simeq \mathcal{M}_3(\mathbb{C}P^1)$ only when $g \geq 6$.*

Proof. For the first statement, Proposition 3 takes care of $g \geq 6$. For $g = 4, 5$, Clifford's Theorem suffices for the conclusion since $6 \leq 2g - 2$.

The case $g = 3$ is finished by Proposition 7. We are left with showing that $\mathcal{M}_3(\mathbb{C}P^1)$ is not isomorphic to $\mathcal{M}_6(M)$ for $3 \le g \le 5$. it is enough to exihbit a g_6^1 which does not come from $H^0(2, H^0(L))$, where $L = (\phi^*\mathcal{O}(\infty))^{\ni}$ with ϕ the branched double covering onto $\mathbb{C}P^1$. To this end, observe first of all that a $g_6^1 \in G(2, H^0(L))$ gives rise to a meromorphic function h of degree 6 on M of the form $f \circ \phi$, where f is meromorphic of degree 3 on $\mathbb{C}P^1$, so that the polar divisor $(h)_\infty$ of h is invariant under the involution τ of M. Now pick a non-Weierstrass point p such that $p \neq \tau(p)$ and consider the divisor $6p$. The Weierstrass gap values at p are $(1,2,3), (1,2,3,4), (1,2,3,4,5)$ for $g = 3, 4, 5$, respectively. It follows that there are meromorphic functions of order 6 whose only pole is p. Take such a function. Then its polar divisor is $6p$, which is not invariant under τ. Hence this function is not of the form $f \circ \phi$ for any f that is rational of degree 3 over $\mathbb{C}P^1$. $\square$

We will classify $\mathcal{M}_6(T)$ for a torus T in Section 6. Contrary to the case $g \ge 3$, lots of nontotally geodesic branched superminimal tori exist.

5. Moduli space of large degree. In contrast with small degrees, we will next show that when the degree d is sufficiently large nontotally geodesic branched superminimal immersions of genus $g \ge 1$ are abundant.

Recall that given a Riemann surface M of genus $g \ge 2$ ($g \ge 1$, respectively) and $d \ge g + 3$ (≥ 3, respectively), M is rationally equivalent to a curve of degree d with at most ordinary nodes as singularities. This follows from the fact that for $g \ge 2$, there exists a nonspecial very ample line bundle of degree d if $d \ge g + 3$ ([11]).

Before proving the existence of a branched superminimal immersion of a sufficiently large degree d into S^4, we recall that a branched superminimal immersion assumes the parametric form

$$[1 : y - 2^{-1}x dy/dx : x : 2^{-1}dy/dx], \tag{12}$$

where x and y are arbitrary meromorphic functions on the Riemann surface ([2]). Notice that one can interpret $[1 : x : y]$ as an algebraic curve in $\mathbb{C}P^2$.

In the following lemma, $o_p(f)$ denotes the pole order of a meromorphic function f at p.

Lemma 7 *Let $F = [1 : x : y]$ be a plane curve with dual curve F^*. Then the branched superminimal immersion G given in (12) is of degree equal to*

$$deg(x) + deg(F^*) - \sum_{p \in M} (\epsilon(p) + \eta(p) + \theta(p) + \zeta(p)),$$

where

$$\epsilon(p) = \max(o_p(y - xdy/dx), 0) \ \ if \ \ o_p(x) = o_p(y),$$
$$\eta(p) = o_p(y) - \max(o_p(y - 2^{-1}xdy/dx) - o_p(x), 0) \ \ if \ \ o_p(y) = 2o_p(x),$$
$$\theta(p) = o_p(x) \ \ if \ \ o_p(x) < o_p(y) \ \ and \ \ o_p(y) \neq 2o_p(x),$$
$$\zeta(p) = o_p(y) \ \ if \ \ o_p(x) > o_p(y),$$
$$\epsilon(p), \eta(p), \theta(p), \zeta(p) = 0 \ \ elsewhere.$$

Proof. We know

$$F^* = [1 : xdy/dx - y : dy/dx].$$

If dy/dx is identically zero, the lemma is trivially true. Assume therefore that $dy/dx \neq 0$. We will count the number of points of intersection of G (F^*, respectively) and the plane $P_1 = \{[s : t : u : 0]\}$ (the plane $P_2 = \{[s : t : 0]\}$, respectively). Let $\sigma(p)$ be the difference between the intersection multiplicities of $G \cap P_1$ and $F^* \cap P_2$ at p.

Case(1). $x = a_0 + a_1 z^\alpha + \cdots$ and $y = b_0 + b_1 z^\beta + \cdots$ around $z = 0$ identified with $p \in M$. Then dy/dx is a zero of order $\beta - \alpha$ at p (if $\beta > \alpha$ of course). All the other coordinate functions for F^* and G are holomorphic around $z = 0$. Hence $\sigma(p) = 0$.

Case(2). $x = a_0 + a_1 z^\alpha + \cdots$ and $y = z^{-\beta} + b_1 z^{-\beta+1} + \cdots$. Then p is a pole of order $\alpha + \beta$ for dy/dx, and all other coordinate functions for F^* and G have poles of order $\leq \alpha + \beta$. In other words, $P_1 \cap G$ and $P_2 \cap F^*$ are empty at p, and so $\sigma(p) = 0$.

Case(3). $x = z^{-\alpha} + a_1 z^{-\alpha+1} + \cdots$ and $y = b_0 + b_1 z^\beta + \cdots$. Then p is a zero of order $\alpha + \beta$ for dy/dx. The second coordinate functions for both F^* and G are holomorphic around $z = 0$, whereas x, having a pole of order α at p, contributes α to the intersection multiplicity of $G \cap P_1$. Hence $\sigma(p) = o_p(x)$.

Case(4). $x = z^{-\alpha} + a_1 z^{-\alpha+1} + \cdots$ and $y = z^{-\beta} + b_1 z^{-\beta+1} + \cdots$. Then G is of the form $[1 : (1 - \beta/2\alpha) z^{-\beta} : z^{-\alpha} : (\beta/\alpha) z^{\alpha-\beta}]$ and F^* of the form $[1 : (1 - \beta/\alpha) z^{-\beta} : (\beta/\alpha) z^{\alpha-\beta}]$. Then (a): If $\alpha = \beta$, then $G \cap P_1$ is of intersection multiplicity α while $F^* \cap P_2$ is of intersection multiplicity $o_p(y - xdy/dx)$ at p, which is $\leq \alpha$. Hence $\sigma(p) = o_p(x) - \epsilon(p)$. (b): If $\beta = 2\alpha$, then the intersection multiplicity of $F^* \cap P_2$ is α while the intersection multiplicity of $G \cap P_1$ is $max(o_p(y - 2^{-1}xdy/dx) - \alpha, 0)$. Hence $\sigma(p) = o_p(x) - \eta(p)$. (c): If $\alpha < \beta$ and $\beta \neq 2\alpha$, then both $G \cap P_1$ and $F^* \cap P_2$ have intersection multiplicity equal to α. Hence $\sigma(p) = o_p(x) - \theta(p)$. (d): If $\alpha > \beta$, then $G \cap P_1$ is of intersection multiplicity $2\alpha - \beta$ while $F^* \cap P_2$ is of intersection multiplicity α. Hence $\sigma(p) = o_p(x) - \zeta(p)$. Adding $\sigma(p)$ in the four cases gives the result. □

Remark 3 *It is important to understand the geometric contents of this lemma. In* $\mathbb{C}P^2$*, pick any three independent points* A, B, C *and set up the coordinate system such that* $A = [1:0:0], B = [0:1:0], C = [0:0:1]$*. Given a Riemann surface and a holomorphic map* $f : M \longrightarrow \mathbb{C}P^2$*, the projection with center* C *(*B*, respectively) onto the line* $\overline{AB}$ *(*$\overline{AC}$*, respectively) gives the meromorphic function* x *(*y*, respectively). The cases in Lemma 7 can be rephrased as follows:*

(i) *Case(1) holds if* $f(p) \in \mathbb{C}P^2 \setminus \overline{BC}$*, the affine part of* $\mathbb{C}P^2$*.*

(ii) *Case(2) holds if* $f(p) = C$ *and* $f(M)$ *is transversal to* $\overline{BC}$*.*

(iii) *Case(3) holds if* $f(p) = B$ *and* $f(M)$ *is transversal to* $\overline{BC}$*.*

(iv) *Case(4.a) holds if* $f(p) \in \overline{BC}$ *and* $f(p) \neq B, C$*. Case(4.b) and Case(4.c) hold if* $f(p) = C$ *and* $f(M)$ *is tangent to* $\overline{BC}$*. Case(4.d) holds if* $f(p) = B$ *and* $f(M)$ *is tangent to* $\overline{BC}$*.*

Corollary 1 *Notation as in Lemma 7 and Remark 3, let* M *be rationally equivalent to* $f(M)$*.*

(i) *If* $f(M)$ *does not pass through the points* B *and* C*, and if* $\overline{BC}$ *intersects* $f(M)$ *transversally, then* $deg(G) = deg(F) + deg(F^*)$*.*

(ii) *If* $f(M)$ *does not pass through the points* B, C*, and if* $\overline{BC}$ *intersects* $f(M)$ *transversally with the exception of one generic point to which* $\overline{BC}$ *is tangent, then* $deg(G) = deg(F) + deg(F^*) - 1$*.*

(iii) *If* $f(M)$ *is through* C *but not through* B*, and if* $\overline{BC}$ *intersects* $f(M)$ *transversally except for one generic point different from* C *to which* $\overline{BC}$ *is tangent, then* $deg(G) = deg(F) + deg(F^*) - 2$*.*

(iv) *If* $\overline{BC}$ *is tangent to* $C \in f(M)$ *as a generic tangent line, and if* $\overline{BC}$ *is transversal to* $f(M)$ *otherwise, then* $deg(G) = deg(F) + deg(F^*) - 3$*.*

Proof. (i) is true since it is Case(4a) in Lemma 7 with $(\alpha, \beta) = (1, 1)$ for any point of intersection of $\overline{BC}$ and $f(M)$. Hence $\epsilon(p) = \eta(p) = \theta(p) = \zeta(P) = 0$ and $deg(x) = deg(F)$.

(ii) holds since it is Case(4a) with $(\alpha, \beta) = (1, 1)$ for $deg(F) - 1$ points of intersection at which $\overline{BC}$ intersects $f(M)$ transversally, where $\epsilon(p) = \eta(p) = \theta(p) = \zeta(p) = 0$. Moreover, it is Case(4a) for the point of tangency at which $(\alpha, \beta) = (2, 2)$, where $\epsilon(p) = 1, \eta(p) = \theta(p) = \zeta(p) = 0$. We have $deg(x) = deg(F)$ in this case.

(iii) holds since it is (ii) above at all points of intersection of $\overline{BC}$ and $f(M)$ other than C. At C, it is Case(2) in Lemma 7 with $\epsilon(p) = \eta(p) = \theta(p) = \zeta(p) = 0$. Furthermore, $deg(x) = deg(F) - 1$ since C is the projection center of x and $C \in f(M)$.

(iv) holds since it is (i) above for all points of intersection of $\overline{BC}$ and $f(M)$ other than C. At C, it is Case(4b) with $(\alpha, \beta) = (1, 2)$, where $\eta(p) = 2, \epsilon(p) = \theta(p) = \zeta(p) = 0$. Moreover, $deg(x) = deg(F) - 1$ in this case for the same reason as in (iii). □

Lemma 8 *Notation is as in (6). For a fixed $x \in G^1_d$, there are only finitely many $y \in G^1_d$ for which $\mathcal{RAM}(x) = \mathcal{RAM}(y)$.*

Proof. Let $L_1 = \pi(x)$ and $L_2 = \pi(y)$. $\mathcal{RAM}(x) = \mathcal{RAM}(y)$ implies $K \otimes (L_1)^2 = K \otimes (L_2)^2$, and hence $(L_1)^2 = (L_2)^2$. So there are only finitely many such L_2. Now apply Lemma 2. □

In the following theorem we refer to a Riemann surface of genus g as being "generic" if G^1_d is an irreducible variety of dimension equal to the Brill-Noether number $2d - g - 2$. For example, all Riemann surfaces are generic in this sense if $d \geq 2g - 1$, or $d \geq 2g - 2$ since G^1_d is the canonical blowup of $W^1_d \simeq J(M)$ at the canonical bundle K regarded as a point in J(M) ([1]), or when M is sufficiently general in the moduli space of Riemann surfaces of genus g so that the Brill-Noether Theory applies.

Theorem 4 *Let M be a Riemann surface of genus $g \geq 2$ ($g = 1$, respectively). If $d \geq 5g + 4$ (≥ 6, respectively), then there is a nontotally geodesic branched, generically one-to-one superminimal immersion of degree d from M to S^4. Furthermore, when M be a generic Riemann surface of genus g in the above sense, the dimension of each irreducible component of $\mathcal{M}_d(M)$ is between $2d - 4g + 4$ and $2d - g + 4$, where the upper bound is achieved by the totally geodesic component.*

Proof. Pick a palne curve F of degree $d_1 \geq g + 3$ (≥ 3 if $g = 1$) with only δ nodes as singularities. By the Plücker formula, $g = (d_1 - 1)(d_1 - 2)/2 - \delta$. Let d_2 be the degree of the dual curve of F; we have $d_2 = d(d - 1) - 2\delta$. Thus $d_1 + d_2 = 2g + 3d_1 - 2 \geq 5g + 7$ (≥ 9 if $g = 1$) and any two consecutive $d_1 + d_2$ differ by 3. Now Corollary 1 implies that any such $d_1 + d_2$ and the two numbers between two consecutive $d_1 + d_2$ are achieved as the degree of a nontotally geodesic branched superminimal immersion. Consequently

$5g+4$ (6 if $g=1$) is the first degree that occurs as the degree of a nontotally geodesic branched superminimal immersion in this procedure. That the immersion is generically one-to-one follows easily by inspecting (12).

For the dimension count, a glance at (6) shows that we need to impose the condition $\mathcal{RAM}(x)=\mathcal{RAM}(y)$ for $(x,y)\in G^1_d\times G^1_d$ to find the dimension of $\mathcal{N}_d(M)$. Now since $\mathcal{RAM}$ maps G^1_d to $S^{2g-2+2d}(M)$, the $(2g-2+2d)$-fold symmetric product of M, $\mathcal{RAM}(x)=\mathcal{RAM}(y)$ imposes at most $2g-2+2d$ conditions to carve out a subvariety of $G^1_d\times G^1_d$ of dimension $4d-2g-4$. Hence the set $S=\{(x,y)\in G^1_d\times G^1_d : \mathcal{RAM}(x)=\mathcal{RAM}(y)\}$ is of dimension $\geq (4d-2g-4)-(2g-2+2d)=2d-4g-2$. Notice that both x and y give rise to 3-dimensional meromorphic functions, respectively. So $\dim\mathcal{N}_d(M)\geq (2d-4g-2)+6$, which is the lower bound. Here, we do not need to worry about the other two conditions, namely, $\pi(x)=\pi(y)$ and x and y have disjoint base loci as given in (6), since once we are given an $(x_0,y_0)\in S$ satisfying the two extra conditions, then any element (x,y) in the irreducible component of S containing (x_0,y_0) will satisfy $\pi(x)=\pi(y)$, by continuity, due to Lemma 8. Moreover, for (x,y) near (x_0,y_0), x and y will have disjoint base loci, by continuity again. To obtain the upper bound, observe that for each $x\in\pi^{-1}(L)$ with $L\in W^1_d$, there are only finitely many (x,y) in S by Lemma 8. Hence $\dim S\leq 2d-g-2$, and so $\dim\mathcal{N}_d(M)\leq(2d-g-2)+6$, which is the upper bound. $\square$

Remark 4 *When $g=0$, the upper bound and the lower bound of* $\dim\mathcal{M}_d(M)$ *in Theorem 4 are identical. Hence $\mathcal{M}_d(M)$ is of pure dimension $2d+4$, which is obtained in [12], [17]. When $g=1$, we will show in Section 6 that the lower bound is achieved for $d=6$.*

The lower bound for the degree d in Theorem 4 is sharp when $g=1$, as we will show in Proposition 9 that all the branched superminimal immersions of degree ≤ 5 are totally geodesic if $g=1$. However, it is not sharp for $g\geq 2$. For example, the above lower bound is 14 when $g=2$. Now take a plane quartic curve F of genus 2 with a simple cusp of multiplicity 2 ([15]). The Plücker formula shows that deg(F^*) $=9$ so that deg(G) $=13$. (Notation is as in Corollary 1.) Hence Corollary 1 infers that 10 is a better bound.

We now look at Theorem 4 from a different viewpoint, which will facilitate the calculations to follow in the next section. Recall the map $\mathcal{R}: G(2,H^0(L))\longrightarrow\mathbb{P}(H^0(K\otimes L^2))$ defined in (7). Let $x=[e_1\wedge e_2]$ and $y=[e_3\wedge e_4]$ satisfy $\mathcal{R}(x)=\mathcal{R}(y)$. Then $[e_1\wedge e_2-e_3\wedge e_4]$ is the projection center of $\mathcal{R}$ restricted to $G(2,V_4)$, where V_4 is spanned by $e_1,\cdots,e_4$. Observe that $\omega=e_1\wedge e_2-e_3\wedge e_4$ satisfies $\omega\wedge\omega\wedge\omega=0$. Conversely, a skew-symmetric form ω satisfying $\omega \wedge \omega \wedge \omega = 0$ is either of rank 2 of the form $e_1 \wedge e_2$, or is of rank 4 of the form $e_1 \wedge e_2 - e_3 \wedge e_4$. It is now clear that each point ω in the intersection $\mathcal{I}$ of $Ker(\mathcal{R})$ and the projective variety $\mathcal{Z} = \mathbb{P}(\{\omega \in \bigwedge^2(H^0(L)) : \omega \wedge \omega \wedge \omega = 0\})$ is the center of the restriction of $\mathcal{R}$ to $G(2, V_4)$ for some 4-dimensional linear system V_4 spanned by some e_1, e_2, e_3, e_4. (By Lemma 1, this intersection cannot contain a form ω of rank 2.) Then $f_1 = [e_1 : e_2]$ and $f_2 = [e_3 : e_4]$ give rise to a superminimal immersion. Now since $\dim \mathcal{Z} = 4k - 11$ if $\dim H^0(L) = k$, a simple dimension count says that $\dim \mathcal{I} \geq 2d - 5g - 6$. In particular, $\mathcal{I}$ is nonempty for every $d \geq (5g + 6)/2$. (See [19] for a better bound for a general Riemann surface.) Varying $L \in J(M)$, we must add $g = \dim J(M)$ to the lower bound, which again gives the lower bound in Theorem 4.

Remark 5 *Note, however, that the above construction does not supersede Theorem 4, because elements f_1 and f_2 which come from $\mathcal{I}$ might have common base loci so that the degree would be lower than d. What Theorem 4 implies is that for a sufficiently large d, there is always a line bundle L of degree d for which $Ker(\mathcal{R}) \cap \mathcal{Z}$ contains an element $e_1 \wedge e_2 - e_3 \wedge e_4$ such that $e_1, \cdots, e_4$ have disjoint base loci. In any event, the above construction does show the existence of nontotally geodesic branched superminimal immersion of degree $\leq (5g + 7)/2$.*

6. The case when $g = 1$. Let M be a Riemann surface of genus 1 and let L be a line bundle of degree d. Then L is the bundle associated with the divisor (dp) for some point p. By applying the translation $p \longmapsto 0$ on the torus, we may assume without loss of generality that p is 0, so that $H^0(L)$ is generated by the d sections $1, \wp, \wp', \wp'', \cdots, \wp^{(d-2)}$.

Proposition 8 *Let M be a Riemann surface of genus 1 and let L be a line bundle over M of degree $d \leq 5$. Then $\mathcal{R} : \bigwedge^2(H^0(L)) \longrightarrow H^0(K \otimes L^2) = H^0(L^2)$ is injective. Hence the moduli space $\mathcal{M}_d(M)$ consists only of totally geodesic branched superminimal immersions.*

Proof. Recall the notation in (7). Observe that each of $[[1, \wp^{(i)}]]$ and $[[\wp^{(i)}, \wp^{(j)}]], 0 \leq i, j \leq d - 2$, consists only of all even or all odd order terms in the pole part of its Laurent expansion. If $d \leq 4$, then an easy computation shows that the orders of the leading terms in the Laurent expansions of $[[1, \wp^{(i)}]]$ and $[[\wp^{(i)}, \wp^{(j)}]], 0 \leq i, j \leq d - 2$, are all different; thus these bracketed quantities are independent in $H^0(L^2)$. So $\mathcal{R}$ is injective. For the case

$d = 5$, one checks similarly that those brackets with odd order terms are independent. The only possibility that $\mathcal{R}$ might have a kernel would be resulted from nontrivial linear relations among $[[1, \wp']], [[1, \wp''']], [[\wp, \wp']], [[\wp, \wp'']]$ and $[[\wp, \wp''']]$. Differentiating the wellknown differential equation $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$ sufficiently many times, we obtain

$$\begin{pmatrix} \wp'' \\ \wp'''' \\ \wp\wp'' - (\wp')^2 \\ \wp\wp'''' - \wp'\wp''' \\ \wp'\wp''' - (\wp'')^2 \end{pmatrix} = \begin{pmatrix} -g_2/2, & 0, & 6, & 0, & 0 \\ -12g_3, & -18g_2, & 0, & 120, & 0 \\ g_3, & g_2/2, & 0, & 2, & 0 \\ 0, & 0, & -6g_2, & 0, & 72 \\ -(g_2)^2/4, & -12g_3, & -6g_2, & 0, & 12 \end{pmatrix} \begin{pmatrix} 1 \\ \wp \\ \wp^2 \\ \wp^3 \\ \wp^4 \end{pmatrix}.$$

It is clear that $1, \wp, \wp^2, \wp^3, \wp^4$ are linearly independent, and a straightforward calculation shows that the determinant of the above 5×5 matrix is $-27 \times 2^{10}((g_2)^3 - 27(g_3)^2) \neq 0$ for a torus. Therefore $\mathcal{R}$ is injective. □

Corollary 2 *For $d \geq 5$, $\mathcal{R}$ is surjective.*

Proof. Since $\dim \bigwedge^2(H^0(L)) = \dim H^0(L^2)) = 10$ if L is of degree 5, Proposition 8 shows that $\mathcal{R} : \bigwedge^2(H^0(L)) \longrightarrow H^0(L^2)$ is bijective. If L is of degree 6, then among the brackets $[[1, \wp^{(i)}]]$ and $[[\wp^{(i)}, \wp^{(j)}]], 0 \leq i, j \leq 4$, the two brackets $[[\wp'', \wp'''']]$ and $[[\wp''', \wp'''']]$ are independent of each other and of all the other brackets since the leading terms in their Laurent expansions are of order 11 and 12, respectively, while others have order ≤ 10. Hence the dimension of the linear subspace generated by all the above brackets in $H^0(L^2)$ is of dimension at least 12, i.e., $\dim \mathcal{R}(\bigwedge^2(H^0(L)) \geq 12$; therefore it is equal to 12 since $\dim \mathcal{R}(\bigwedge^2(H^0(L)) \leq \dim H^0(L^2) = 12$. in other words, $\mathcal{R}$ is surjective. Exactly the same reasoning takes care of all $d \geq 6$. □

It should be mentioned that Corollary 2 had been proved in [18]. However, our proof is elementary.

Now since the projective codimension of $Ker(\mathcal{R})$ is precisely $2d + g - 1 = 2d$ if L is of degree d by Corollary 2, a glance at the construction of $\mathcal{I}$ suggests that $\dim \mathcal{I} = 2d - 5g - 6 = 2d - 11$. We will show next that this is true if $d = 6$.

Proposition 9 *Let M be a torus and L be a line bundle of degree 6. Then $\dim \mathcal{I} = 1$. Hence the nontotally geodesic irreducible components of $\mathcal{M}_6(M)$ all have dimension equal to 12; in particular the lower bound of $\mathcal{M}_d(M)$ in Theorem 4 is achieved by these components.*

Proof. Let $e_1, e_2, \cdots, e_6$ be a basis of $H^0(L)$. The Euclidean dimension of $Ker(\mathcal{R})$ is 3. Let E_1, E_2, E_3 be a bassi of $Ker(\mathcal{R})$; E_1, E_2, E_3 are linear combinations of $e_i \wedge e_j, 1 \leq i, j \leq 6$.

Let $\omega = xE_1 + yE_2 + zE_3, x, y, z \in \mathbb{C}$ be any element in $\mathcal{I}$. Rewriting $\omega \wedge \omega \wedge \omega$ as a multiple of $e_1 \wedge e_2 \wedge \cdots \wedge e_6$ and incorporating the fact that ω satisfies $\omega \wedge \omega \wedge \omega = 0$, we see that ω is defined by a nonvoid homogeneous polynomial of degree 3 in x, y, z. In other words, $\mathcal{I}$ is defined by a plane cubic curve. Hence $\dim \mathcal{I} = 1$. □

When $g = 0$, the nontotally geodesic part of $\mathcal{M}_d(\mathbb{C}P^1)$ is irreducible, and hence $\mathcal{M}_d(\mathbb{C}P^1)$ consists of two irreducible components ([12], [17]). This is not the case in general when $g \geq 1$ as the following proposition shows.

Proposition 10 *Let T be a torus. The nontotally geodesic part of $\mathcal{M}_6(T)$ may be reducible, although for a generic torus it is irreducible.*

Proof. It suffices to find a torus for which $\mathcal{I}$ is reducible and one for which $\mathcal{I}$ is irreducible. Consider the torus with $g_2 = 0$ and $g_3 = 1$. Set $e_1 = 1$ and $e_i = \wp^{(i-2)}, 2 \leq i \leq 6$. Recall that a linear relation $\sum_{i,j} x_{ij}[[e_i, e_j]] = 0$ gives $\sum_{ij} x_{ij} e_i \wedge e_j$ in $Ker(\mathcal{R})$, and vice versa. Comparing the coefficients in the Laurent expansions of all $[[e_i, e_j]]$ via the identity

$$\wp = 1/z^2 + \sum_{i=1}^{\infty} a_{2i} z^{2i},$$

where

$$a_2 = g_2/20,$$
$$a_4 = g_3/28,$$
$$a_{2(n+1)} = 3(n-1)^{-1}(2n+5)^{-1} \sum_{i=1}^{n-1} a_{2i} a_{2(n-i)}$$

with $n \geq 2$, one ends up with three generators

$$E_1 = 108 e_1 \wedge e_3 + e_3 \wedge e_6 - 5 e_4 \wedge e_5,$$
$$E_2 = 72 e_1 \wedge e_2 - e_2 \wedge e_6 + e_3 \wedge e_5,$$
$$E_3 = -e_1 \wedge e_6 + 60 e_2 \wedge e_4$$

for $Ker(\mathcal{R})$. Let $w = xE_1 + yE_2 + zE_3$. Then $\omega \wedge \omega \wedge \omega = 0$ results in the equation

$$yz^2 - 3x^2 y = 0,$$

which is the union of three lines defining $\mathcal{I}$, and so $\mathcal{I}$ is reducible. On the other hand, setting $g_2 = 1$ and $g_3 = 0$ yields

$$\begin{aligned} E_1 &= 5e_1 \wedge e_4 - e_2 \wedge e_6 + 5e_3 \wedge e_5, \\ E_2 &= -48e_1 \wedge e_2 - e_1 \wedge e_6 + 60e_2 \wedge e_4, \\ E_3 &= -72e_2 \wedge e_3 - e_3 \wedge e_6 + 5e_4 \wedge e_5. \end{aligned}$$

Hence $\omega \wedge \omega \wedge \omega = 0$ with $\omega = xE_1 + yE_2 + zE_3$ asserts that

$$-x^3 + 12xy^2 + 24yz^2 = 0,$$

which is the torus with $g_2 = 1$ and $g_3 = 0$ that defines $\mathcal{I}$. Thus $\mathcal{I}$ is irreducible in this case. □

7. Concluding remarks. Propositions 9 and 10 point to the challenging question whether the nontotally geodesic part of $\mathcal{M}_d(M)$ is of pure dimension $2d - g + 4$ for any Riemann surface of genus g, and whether it is irreducible, so that the moduli space of branched superminimal surfaces of degree d consists of three irreducible components, for a generic Riemann surface of genus g.

As for the compactification of $\mathcal{N}_d(M)$, and so for that of $\mathcal{M}_d(M)$, a glance at (6) suggests that the space

$$\overline{\mathcal{N}}_d(M) = \left\{ (f_1, f_2) \in \overline{R}^1_d \times \overline{R}^1_d : \begin{array}{l} \pi\pi_2(f_1) = \pi\pi_2(f_2), \\ \mathcal{R}(f_1) = \mathcal{R}(f_2) \end{array} \right\}$$

is the natural candidate, which Loo adopted in [12] when the genus $g = 0$. Whether this is true in the higher genus case depends on whether $\pi_2(f_1)$ and $\pi_2(f_2)$ having disjoint loci encountered in (6) is a generic condition; for if it is not a generic condition, we will have an irreducible component of $\mathcal{M}_d(M)$ which comprises entirely of branched superminimal immersions of degree lower than d, then $\overline{\mathcal{N}}_d(M)$ will be too big for the compactification.

We suspect that the answers to these questions are all affirmative for a generic Riemann surface of genus g, which would follow if the intersection of $Ker(\mathcal{R})$ and $\mathcal{Z}$ were transversal for all L in $J(M)$.

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