The dimension of the moduli space of superminimal surfaces of a fixed degree and conformal structure in the 4-sphere

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Abstract

It was established by X. Mo and the author that the dimension of each irreducible component of the moduli space $\mathcal{M}_{d,g}(X)$ of branched superminimal immersions of degree d from a Riemann surface X of genus g into $\mathbb{C}P^3$, lay between $2d-4g+4$ and $2d-g+4$ for d sufficiently large, where the upper bound was always assumed by the irreducible component of *totally geodesic* branched superminimal immersions and the lower bound was assumed by all *nontotally geodesic* irreducible components of $\mathcal{M}_{6,1}(T)$ for any torus T. It is shown, via deformation theory, in this note that for $d = 8g + 1 + 3k, k \geq 0$, and any Riemann surface X of $g \geq 1$, the above lower bound is assumed by at least one irreducible component of $\mathcal{M}_{d,g}(X)$.

0. The dimension and irreducibility are two fundamental questions when dealing with moduli spaces. In [2] Calabi studied minimal 2-spheres in an ambient round sphere, where he showed that the ambient sphere must be of even dimension if the minimal 2-sphere is linearly full in the ambient sphere. Moreover, all the minimal 2-spheres are obtained by projecting horizontal holomorphic rational curves from the twistor space of the ambient sphere S^{2n} into S^{2n}. Here, the twistor space of S^{2n} is the Hermitian symmetric space of pointwise orthogonal complex structures of S^{2n}, and horizontality refers to the horizontal distribution of the twistor space naturally induced by the Riemannian connection of S^{2n}. In general, the projection of any

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horizontal holomorphic curve from the twistor space into S^{2n} is a minimal surface called a (branched) *superminimal* surface.

The twistor space of S^4 happens to be the pleasant $\mathbb{C}P^3$, where a horizontal holomorphic curve satisfies the differential equation

$$z_0 dz_1 - z_1 dz_0 + z_2 dz_3 - z_3 dz_2 = 0 \tag{1}$$

with the homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$ of $\mathbb{C}P^3$, with respect to which Bryant [1] proved the existence of branched superminimal surfaces of arbitrary genus and conformal structure in S^4. Loo [10] and Verdier [12] later studied the moduli space of the branched superminimal spheres of a fixed area (equal to a constant multiple of the degree d of the corresponding horizontal holomorphic curves). Subsequently Mo and I [3] investigated the moduli space $\mathcal{M}_{d,g}(X)$ of branched superminimal surfaces of a fixed degree d from any Riemann surface X of genus g into the four-sphere. By definition $\mathcal{M}_{d,g}(X)$ is the variety of all horizontal holomorphic maps from X into $\mathbb{C}P^3$ satisfying (1). From this equation one sees that $\mathcal{M}_{d,g}(X)$ is, roughly, a double cover of the variety $\mathcal{N}_{d,g}(X)$ consisting of pairs of meromorphic functions (f, g) over X such that f and g have equivalent polar divisors and identical ramification divisors (see [3] for more details). For the Riemann sphere, the non-diagonal part of $\mathcal{N}_{d,g}(X)$, i.e., the set of elements not of the form $(A \circ f, B \circ f)$ with A and B being Möbius transformations, which corresponds geometrically to the set of nontotally geodesic branched superminimal immersions, is always irreducible by a result of Einsenbud and Harris [4], as Loo and Verdier pointed out. However, Mo and I exhibited a certain torus T of degree 6 for which the the non-diagonal part of $\mathcal{N}_{6,1}(T)$ is *not* irreducible [3].

As for the dimension of the moduli space, or equivalently of $\mathcal{N}_{d,g}(X)$, we showed in [3] that although for a small degree the conformal structure of such a nontotally geodesic branched superminimal surface is very restricted, for any sufficiently large degree nontotally geodesic branched superminimal surfaces do exist for any conformal structure and the dimension of each irreducible component of the moduli space is between $2d-4g+4$ and $2d-g+4$. Setting $g=0$, one sees that the moduli space for the Riemann sphere is therefore of pure dimension $2d+4$ proved by Loo and Verdier. The upper bound $2d-g+4$ is always achieved by the branched totally geodesic superminimal surfaces, or equivalently by the diagonal part of $\mathcal{N}_{d,g}(X)$, and the lower bound $2d - 4g + 4$ is achieved by all non-diagonal irreducible components of $\mathcal{N}_{6,1}(T)$ for any torus T.

It is tempting to suspect that the non-diagonal part of $\mathcal{N}_{d,g}(X)$ is of pure dimension $2d - 4g + 4$ for all X as long as d is sufficiently large.

From a different angle, the above equation defines the canonical *contact* structure of $\mathbb{C}P^3$. Recall that by a complex contact 3-fold W we mean there endows on W a holomorphic line bundle L^* of 1-forms such that if θ is a local section of L^* (called a local contact form), then $\theta \wedge d\theta$ is a nondegenerate 3-form. The dual of L^* in TW is the 2-dimensional contact distribution $\mathcal{D}$, with respect to which L, the dual of L^* called the contact line bundle of W, is isomorphic to $TW/\mathcal{D}$. A transition function computation [7] gives that

$$L^{-2} = \mathcal{K}, \tag{2}$$

where $\mathcal{K}$ is the canonical bundle of W. By Darboux's theorem, there is a local coordinate system (p, q, r) relative to which the local contact form can be written as

$$\theta = dr + pdq - qdp. \tag{3}$$

In fact, (1) comes down to (3) in affine coordinates of $\mathbb{C}P^3$ when one sets one of the homogeneous coordinates equal to 1. Note that by (2)

$$L = \mathcal{O}(2)$$

for $\mathbb{C}P^3$.

Hence, the moduli space $\mathcal{M}_{d,g}(X)$ intuitively may be thought of as a "family" of *contact* maps from X into $\mathbb{C}P^3$ (i.e., maps whose images are curves tangent to the contact distribution $\mathcal{D}$). Utilizing this second approach, we will prove in this note the following.

Theorem 1 *Let $d = 8g + 1 + 3k, k \geq 0$, for any Riemann surface X of genus $g \geq 1$. Then the dimension of at least one irreducible component of each $\mathcal{M}_{d,g}(X)$ achieves the above lower bound $2d - 4g + 4$.*

We first make precise in the next section the notion of a family of contact maps from Riemann surfaces into a contact 3-fold W (whose images may be highly singular contact curves with varying conformal structures), and find conditions for the existence and completeness of such a family. We then specialize to $\mathbb{C}P^3$ for the conclusion of the theorem.

1. Recall that by a family $(\mathcal{F}, \Phi, p, \mathcal{M})$ of holomorphic maps into a complex manifold W we mean complex manifolds $\mathcal{F}$ and $\mathcal{M}$ and two holomorphic maps $p : \mathcal{F} \longrightarrow \mathcal{M}$ and $\Phi : \mathcal{F} \longrightarrow W \times \mathcal{M}$ such that (1) p is a holomorphic submersion such that $p^{-1}(t)$ is connected for all t in $\mathcal{M}$, and (2) $\Pi \circ \Phi = p$, where $\Pi : W \times \mathcal{M} \longrightarrow \mathcal{M}$ is the projection [6], [9], [11]. We call $\mathcal{F}$ the total space, $\mathcal{M}$ the base space and Φ the total deformation map. Intuitively, we think of $\mathcal{M}$ as the parameter space locally parametrized by

$$t = (t_1, \cdots, t_\alpha, \cdots, t_n).$$

For notational ease, we will not distinguish the Euclidean coordinates of a manifold from its corresponding manifold neighborhood henceforth. Sitting over each t is a complex manifold $X_t = p^{-1}(t)$ which is mapped to W by the map $f_t = \Phi|_{X_t}$ followed by the projection onto the first factor of $W \times \mathcal{M}$.

For us W will be a contact 3-manifold, X_t will be Riemann surfaces of genus g, whose conformal structures may vary, and $f_t : X_t \longrightarrow W$ will be nontrivial *contact* maps in the sense that $f_*(TX_t)$ is tangent to the contact distribution of W. Although the image of X_t may be highly singular curves, the singularities occur only at finitely many points. Hence we always have the exact sequence

$$0 \longrightarrow T_{X_t} \longrightarrow f_t^*(TW) \longrightarrow N_t \longrightarrow 0, \tag{4}$$

where N_t is the cokernel, for all t.

Let $\mathcal{D}$ be the contact distribution of W and let $L := TW/\mathcal{D}$ be the contact line bundle. We have the exact sequence

$$0 \longrightarrow f_t^*(\mathcal{D}) \longrightarrow f_t^*(TW) \longrightarrow f_t^*L \longrightarrow 0. \tag{5}$$

We wish to understand the tangent spaces of the deformation family. Recall that $\mathcal{M}$ is locally parametrized by $(t_1, \cdots, t_\alpha, \cdots, t_n)$, which we may assume contains 0. Let us cover $\mathcal{F}$ with coordinates $U_i = \{(z_i, t)\}$ over a neighborhood of $0 \in \mathcal{M}$ such that (z_i) cover X_0 and $\Phi_i := \Phi|_{U_i} : U_i \longmapsto W_i$, where $W_i = \{(p_i, q_i, r_i)\}$ form a contact coordinate cover of W in the sense that over W_i the contact form may be chosen to be

$$dr_i + p_i dq_i - q_i dp_i, \tag{6}$$

so that we have

$$\Phi_i : (z_i, t) \longmapsto (p_i(z_i, t), q_i(z_i, t), r_i(z_i, t))$$

with

$$\frac{\partial r_i}{\partial z_i} + p_i \frac{\partial q_i}{\partial z_i} - q_i \frac{\partial p_i}{\partial z_i} = 0. \tag{7}$$

We choose the coordinates z_i so small that a singular point of f_0, that is, a point of X_0 where $df_0 = 0$, is placed at the origin of only one such a coordinate.

In view of (4) and (5), for each α the collection

$$s_{i,\alpha} := \frac{\partial r_i}{\partial t_\alpha} + p_i \frac{\partial q_i}{\partial t_\alpha} - q_i \frac{\partial p_i}{\partial t_\alpha}\Big|_{t=0} \tag{8}$$

defines a holomorphic section s_α of $f_0^* L$. Differentiating $s_{i,\alpha}$ with (7) in mind, we obtain

$$\frac{ds_{i,\alpha}}{dz_i} = \frac{\partial q_i}{\partial t_\alpha}\frac{\partial p_i}{\partial z_i} - \frac{\partial p}{\partial t_\alpha}\frac{\partial q_i}{\partial z_i}\Big|_{t=0}. \tag{9}$$

At a point x covered by coordinate z_i on X_0, we denote by $o(x)$ the minimum of the vanishing order of the three functions $\partial p_i/\partial z_i, \partial q_i/\partial z_i, \partial r_i/\partial z_i$, which is an analytic invariant and is not zero only at singular points of the map f_0. (In fact it suffices to consider only p_i and q_i in view of (7). We define the *singular* divisor of the map f_0 to be

$$\mathcal{S} = \sum_{x \in X_0} o(x).$$

For a section $s = (s_i(z_i))$ of $f_0^* L$, we denote by $(ds)_0$ the divisor of the vanishing order of ds_i/dz_i at the singular points of f_0. Then (9) says that the correspondence

$$\frac{\partial}{\partial t_\alpha} \longmapsto s_\alpha$$

maps $T_0\mathcal{M}$ to

$$\mathcal{T}_0 = \{s \in H^0(f_0^* L) : (ds)_0 \geq \mathcal{S}\}.$$

Now let $s_i(z_i)$ be any local section of $f_0^* L$ such that $(ds_i)_0 \geq \mathcal{S}$. Let $a(z_i)$ and $b(z_i)$ be two (local) holomorphic functions such that

$$\frac{ds_i}{dz_i} = 2b(z_i)\frac{\partial p_i}{\partial z_i}(z_i, 0) - 2a(z_i)\frac{\partial q_i}{\partial z_i}(z_i, 0).$$

Set $c = s_i + q_i a - p_i b$. Then define in $f_0^*(TW)$ the local *contact* vector field

$$V_i := a\frac{\partial}{\partial p_i} + b\frac{\partial}{\partial q_i} + c\frac{\partial}{\partial r_i}.$$

Lemma 1 *Any other choice of $a(z_i)$ and $b(z_i)$ results in a difference in V_i by a change in $\mathcal{O}(TX_0)$. Hence the projection of V_i into $\mathcal{N}_0$ via* (4) gives rise to a well-defined map $s_i \longmapsto [V_i] \in \mathcal{N}_0$.

Proof. Let $a_1(z_i)$ and $b_1(z_i)$ be another choice giving rise to the vector field V_i'. Then

$$(b - b_1)\frac{\partial p_i}{\partial z_i}(z_i, 0) = (a - a_1)\frac{\partial q_i}{\partial z_i}(z_i, 0).$$

We may assume that $q_i' := \partial q_i / \partial z_i(z_i, 0)$ is nowhere zero if z_i parametrizes smooth points, whereas if $z_i = 0$ is a singular point, we assume that the vanishing order of q_i' is no greater than that of $p_i' := \partial p_i / \partial z_i(z_i, 0)$. A calculation derives

$$V_i - V_i' = \frac{b - b_1}{q_i'}[p_i'(\frac{\partial}{\partial p_i} + q_i\frac{\partial}{\partial r_i}) + q_i'(\frac{\partial}{\partial q_i} - p_i\frac{\partial}{\partial r_i})].$$

The conclusion follows when we observe that

$$p_i'\frac{\partial}{\partial p_i} + q_i'\frac{\partial}{\partial q_i} + r_i'\frac{\partial}{\partial r_i} = p_i'(\frac{\partial}{\partial p_i} + q_i\frac{\partial}{\partial r_i}) + q_i'(\frac{\partial}{\partial q_i} - p_i\frac{\partial}{\partial r_i})$$

in view of (7). $\square$

Theorem 2 *Let $f_0 : X_0 \longrightarrow W$ be a holomorphic contact map from a Riemann surface of genus g into a contact* 3-fold with contact line bundle L. Let $\mathcal{S}$ be the singular divisor of f_0 and $[\mathcal{S}]$ the line bundle determined by it. Suppose

(1) $H^1(f_0^*L) = 0$, and

(2) $\pi : H^0(f_0^*L) \longrightarrow H^0([\mathcal{S}]|_{\mathcal{S}})$ *given by*

$$s = (s_i(z_i)) \longmapsto \oplus\frac{\partial s_k}{\partial z_k} \pmod{\mathcal{S}}$$

is surjective, where z_k is the coordinate around a singular point and (mod $\mathcal{S}$) indicates that the Taylor series is truncated modulo the appropriate singular order in $\mathcal{S}$ at the point. Then there is a family of holomorphic contact maps from Riemann surfaces of genus g into W such that f_0 is the initial map and the dimension of the family is

$$1 + \deg(f_0^*L) - g - deg(\mathcal{S}),$$

as long as $\deg(f_0^*L) \geq 2g - 1$.

Proof. (Sketch). Let g_{ij}, where

$$(p_i, q_i, r_i) = g_{ij}(p_j, q_j, r_j),$$

be the transition function of the contact coordinates of W. Let b_{ij} be the transition function of the initial Riemann surface X_0 and f_i be the restriction of the initial map $f_0 : X_0 \longmapsto W$ to the coordinate z_i. Following [6], where the deformation of an arbitrary map is considered, we want to construct formal power series $\phi_{ij}(z_j, t)$ and $\Phi_i(z_i, t)$ such that

$$\begin{aligned} \phi_{ij}(z_j, 0) &= b_{ij}(z_j), \\ \phi_{ij}(\phi_{jk}(z_k, t), t) &= \phi_{ik}(z_k, t), \\ \Phi_i(z_i, 0) &= f_i(z_i), \\ \Phi_i(\phi_{ij}(z_j, t), t) &= g_{ij}(\Phi_j(z_j, t)). \end{aligned} \tag{10}$$

Our case involves one more condition than these four. Namely, $\Phi_i(z_i, t)$ must satisfy (7) as well.

We also adopt Kodaira's convention that for a power series $P(t_1, \cdots, t_n)$, we denote by P^m the finite sum of the series up to the m-th degree, by $P_{|m}$ the term of m-th degree, and by $P \equiv_m Q$ to indicate that the two polynomials P and Q agree up to degree m.

In [6] one solves the polynomial version of the second and the fourth item of (10):

$$\begin{aligned} \phi^m_{ij}(\phi^m_{jk}(z_k, t), t) &\equiv_m \phi^m_{ik}(z_k, t), \\ \Phi^m_i(\phi^m_{ij}(z_j, t), t) &\equiv_m g_{ij}(\Phi^m_j(z_j, t)). \end{aligned}$$

The difference between our case and that in [6], however, is that our deformation must always be contact. To achieve this goal, observe that there is a map

$$\rho : \mathcal{T}_0 \longrightarrow H^1(X_0, \mathcal{O}(TX_0))$$

obtained, from the above lemma, by sending $\mathcal{T}_0$ into $H^0(X_0, \mathcal{N}_0)$ followed by the connecting homomorphism

$$H^0(X_0, \mathcal{N}_0) \longrightarrow H^1(X_0, \mathcal{O}(TX_0))$$

of (4). Now $\rho(\partial/\partial t_\alpha)$ is a 1-cocycle $(\theta^\alpha_{ij}(z_j))$. We set

$$\phi^1_{ij}(z_j, t) := b_{ij}(z_j) + \sum_\alpha \theta^\alpha_{ij}(z_j) t_\alpha,$$

and, in view of (8),

$$\Phi_i^1(z_i,t) := f_i(z_i) + \sum_\alpha s_{i,\alpha}(z_i)t_\alpha.$$

Suppose $\phi_{ij}^{m-1}(z_j,t)$ and $\Phi_i^{m-1}(z_i,t)$, $m \geq 2$, have been determined. Then it is shown in [6] that the collection

$$\Gamma_{ij|m}(z_i,t) = [\Phi_i^{m-1}(\phi_{ij}^{m-1}(z_j,t),t) - g_{ij}(\Phi_j^{m-1}(z_j,t))]_{|m}$$

defines a 1-cocycle in $f_0^*\mathcal{N}_0$, which projects via (5) to a 1-cocycle $(s_{ij|m})$ in f_0^*L. Since $H^1(f_0^*L) = 0$ by assumption (1), we have

$$s_{ij|m} = s_{j|m} - s_{i|m}.$$

The ramification order $(ds_{i|m})_0$ at a singular point of f_0 smaller than the order of the singular divisor at the point can be eliminated by assumption (2), i.e., there is a global section $s_{|m}$ of $f_0^*(L)$ such that the local sections $s'_{i|m} := s_{i|m} - s_{|m}$ satisfies

(11) $$(ds'_{i|m})_0 \geq \mathcal{S},$$

and

$$s_{ij|m} = s'_{j|m} - s'_{i|m}.$$

From (11) and the above lemma, we can find a contact vector field $\Phi_{i|m}$, unique up to $\mathcal{O}(TX_0)$, such that $\Phi_{i|m}$ projects to $s'_{i|m}$. Now we define

$$\Phi^m = \Phi^{m-1} + \Phi_{i|m},$$

which completes the induction. One then goes through Kodaira's argument [8],[9] verbatim to show the convergence of the series $\sum_m \Phi_{|m}$ for sufficiently small t.

Since the dimension of the deformation family is that of $\mathcal{T}_0$, which is the kernel of π, the dimension count follows from Riemann-Roch. □.

In particular, $\mathcal{S} = 0$ when f_0 is an immersion. The deformation family is then of dimension equal to $\dim H^0(f_0^*L) = 1 + \deg(f_0^*L) - g$ as long as $\deg(f_0^*L) \geq 2g - 1$.

We denote by $\mathcal{S} + 1$ the divisor supported at the singular points whose order at a singular point is one more than the singular order there, by $[\mathcal{S}+1]$

the line bundle generated by the divisor, and by $|\mathcal{S}|$ the number of singular points. A sufficient condition for the surjectivity of π, i.e., for assumption (2) in Theorem 2 to hold, is to consider the exact sequence

$$0 \longrightarrow f_0^*L - [\mathcal{S}+1] \longrightarrow f_0^*L \longrightarrow [\mathcal{S}+1]|_{\mathcal{S}+1} \longrightarrow 0.$$

The surjectivity of π will be ensured if $H^1(f_0^*L - [\mathcal{S}+1]) = 0$, which is the case if

$$\deg(f_0^*L) > 2g - 2 + \deg(\mathcal{S}) + |\mathcal{S}|. \tag{12}$$

Hence π is surjective as long as the degree of f_0^*L is much larger than the singular divisor. For instance, when $W = \mathbb{C}P^3$, f_0 is of degree d and so $\deg(f_0^*L) = 2d$ since $L = \mathcal{O}(2)$. The Plücker formula asserts that (12) is equivalent to

$$d_1 > 4g - 4 + |\mathcal{S}|,$$

where d_1 is the degree of the first associated curve of f_0.

Recall [6] that given a family $(\mathcal{F}, \Phi, p, \mathcal{M})$, abbreviated $f_t : X_t \longrightarrow W$, and a holomorphic map h from a manifold $\mathcal{M}'$ to $\mathcal{M}$, there is an *induced* family $(\mathcal{F} \times_{\mathcal{M}} \mathcal{M}', \Phi \times id, p', \mathcal{M}')$, where p' is the natural projection from $\mathcal{F} \times_{\mathcal{M}} \mathcal{M}'$ to $\mathcal{M}'$. The induced family is nothing but the family of maps $f_{h(t)}$.

We say that a family $(\mathcal{F}, \Phi, p, \mathcal{M})$ with the initial map f_0 over $t_0 \in \mathcal{M}$ is *complete* at t_0 if given any other family $(\mathcal{F}', \Phi', p', \mathcal{M}')$ with the initial map g_0 over $t_0' \in \mathcal{M}'$ equivalent to f_0, there is a holomorphic map h from a neighborhood of $t_0' \in \mathcal{M}'$ to that of $t_0 \in \mathcal{M}$ mapping t_0' to t_0 such that the latter family is equivalent to the family induced from the former one. Here two families are said to be equivalent if there exist biholomorphisms between the total and the base spaces respectively that commute with the two total deformation maps.

Theorem 3 *Notation being as above, if $\mathcal{T}_0 = H^0(f_0^*L)$, then the family is complete at $0 \in \mathcal{M}$.*

Proof. (Sketch). Following [6], it suffices to construct a local holomorphic function $h(t')$ from a neighborhood around $0 \in \mathcal{M}'$ to $\mathcal{M}$ and holomorphic functions $g_i(z_i, t')$ such that

$$h(0) \;=\; 0,$$

$$\begin{aligned} g_i(z_i, 0) &= z_i, \\ g_i(\phi'_{ij}(z_j, t'), t') &= \phi_{ij}(g_j(z_j, t'), h(t')), \\ \Phi_i(g_i(z_i, t'), h(t')) &= \Phi'_i(z_i, t'). \end{aligned}$$

Of course, the deformation in our case must always be contact. Again one considers the polynomial version of the system

$$\begin{aligned} g_i^m(\phi'_{ij}, t') &\equiv_m \phi_{ij}(g_j^m, h^m), \\ \Phi_i(g_i^m, h^m) &\equiv_m \Phi'_i. \end{aligned}$$

Suppose that we have constructed h^{m-1} and g_i^{m-1}. As in [6], the quantities

$$\gamma_{i|m} = [\Phi'_i - \Phi_i(g_i^{m-1}, h^{m-1})]_{|m} \cdot \frac{\partial}{\partial w_i}$$

defines a global section of $\mathcal{N}_0$, where $\partial/\partial w_i$ denotes $(\partial/\partial p_i, \partial/\partial q_i, \partial/\partial r_i)$, and $F_i \cdot \partial/\partial w_i$ denotes $\sum_\alpha F_i^\alpha \partial/\partial w_i^\alpha$. In fact one has the identity [6]

$$\gamma_{i|m} = \iota(g_{i|m}) + \sum_\alpha h_{\alpha|m} \frac{\partial \Phi_i}{\partial t_\alpha}, \tag{13}$$

where $\iota : TX_0 \longrightarrow f_0^* TW$ appears in (4) and $h_{|m} = (h_{1|m}, \cdots, h_{\alpha|m}, \cdots)$. The projection of $\gamma_{i|m}$ defines a section $t := (t_{i|m})$ in $H^0(f_0^* L)$ such that

$$t_{i|m} = \sum_\alpha h_{\alpha|m} s_{i,\alpha} \tag{14}$$

with $s_{i,\alpha}$ defined by (8). So (14) determines $h_{|m}$. To find $g_{i|m}$, observe that since now $H^0(f_0^* L) = \mathcal{T}_0$, we can, by the above lemma, find local contact vector fields, unique up to $\mathcal{O}(TX_0)$, which correspond to $t_{i|m}$ and $s_{i,\alpha}$; the injectivity of ι then finishes the work. The same arguments as in [6] will prove the convergence of the power series $\sum_m g_{|m}$ and $\sum_m h_{|m}$ for sufficiently small t. □

In particular, if f_0 is an immersed contact map such that $\deg(f_0^* L) \geq 2g - 1$, then there is a complete deformation family of holomorphic contact maps of dimension $1 + \deg(f_0^* L) - g$ around f_0.

2. We now prove Theorem 1. It is well-known [11] that the family of Riemann surfaces of genus g parametrized by the Teichmüller space T_g can be simultaneously embedded into $\mathbb{C}P^{1+\tau-g}$ as curves of degree τ for $\tau \geq 2g+1$. We will identify the Riemann surfaces with the image curves. Let X_t be a 1-parameter family of such curves with X_0 the initial Riemann surface. Choose a generic projection from $\mathbb{C}P^{1+\tau-g}$ onto $\mathbb{C}P^2$ so that X_t are projected onto plane curves $c(t)$ (of the same degree τ) with only nodes as singularities. In $\mathbb{C}P^2$ pick three independent points A, B, C and set up the projective coordinates with $A = [1:0:0], B = [0:1:0], C = [0:0:1]$ in such a way that for any t in a small neighborhood of $t = 0$ the line BC intersects the curve $c(t)$ transversally, $c(t)$ does not pass through B, C, and all tangent lines of $c(t)$ passing through C have contact order 2 with $c(t)$. The projection with center C (B, respectively) onto the line AB (line AC, respectively) gives rise to meromorphic functions x_t and y_t on X_t. These two meromorphic functions generate immersed holomorphic contact maps $f_t : X_t \longrightarrow \mathbb{C}P^3$ of degree

$$d := 2g + 3\tau - 2$$

([3, Corollary 1]).

Now the connecting homomorphism

$$\Delta_0 : H^0(\mathcal{N}_0) \longrightarrow H^1(TX_0)$$

arising from (4) sends the infinitesimal normal deformation of the 1-parameter contact family f_t exactly to the infinitesimal deformation of the complex structure of X_t at $t = 0$ ([6]), which is a tangent vector of the tangent space of the Teichmüller space at X_0 identified with $H^1(TX_0)$. Thus because the choice of the 1-parameter family is arbitrary, Δ_0 sends the infinitesimal normal deformation of the family $\mathcal{F}$ of immersed contact maps containing f_0, which is a subspace of $H^0(\mathcal{N}_0)$ of dimension $2d - g + 1$, onto $H^1(TX_0)$ of dimension $3g - 3$ if $g \geq 2$ and 1 if $g = 1$. Thus there is a neighborhood U of 0 in the parameter space $\mathcal{M}$ of the family $\mathcal{F}$ such that for any $t \in U$ the connecting homomorphism $\Delta_t : H^0(\mathcal{N}_t) \longrightarrow H^1(TX_t)$ maps the infinitesimal normal deformation of $\mathcal{F}$ at t onto $H^1(TX_t)$, so that the kernel $\mathcal{K}_t$ is of dimension $2d - g + 1 - 3g + 3 = 2d - 4g + 4$ if $g \geq 2$ and $2d - 4g + 3$ if $g = 1$.

Consider the moduli space $\mathcal{M}_{d,g}(X_0)$. Let $\mathcal{V} \subset \mathcal{M}_{d,g}(X_0)$ be the irreducible component containing f_0, and let $\gamma : |z| \leq \epsilon \longmapsto \mathcal{V}$ be a parametrized curve with $\gamma(0) = f_0$ such that $\gamma(\epsilon)$ is a smooth point of $\mathcal{V}$. We may choose ϵ so small that all $\gamma(z)$ are immersed maps, so that $\gamma(z)$ is in fact a family of immersed contact maps. By the completeness of $\mathcal{F}$, the family $\gamma(z)$ is induced from $\mathcal{F}$ so that $\gamma(\epsilon)$ lies in $\mathcal{F}$ and we may assume it is parametrized by some $t^0 \in U$ by choosing ϵ small enough. A sufficiently small neighborhood of $\gamma(\epsilon)$ in $\mathcal{V}$ consists of a family of immersed contact maps whose conformal structures remain fixed, and hence whose infinitesimal normal deformation, which is nothing but the tangent space to $\mathcal{V}$ at f_{t^0}, lies in the kernel $\mathcal{K}_{t^0}$. Therefore for $g \geq 2$, we have $\dim \mathcal{V} \leq 2d - 4g + 4$. However, we have proved in [3] that $\dim \mathcal{V} \geq 2d - 4g + 4$ as mentioned in Section 0. So the equality is attained. For $g = 1$, the same argument would at first glance show $\dim \mathcal{V} \leq 2d - 4g + 3$, which seems to contradict $\dim \mathcal{V} \geq 2d - 4g + 4$. However, there is a 1-dimensional worth of translations on the torus that do not appear in the deformation, and any contact map composed with a translation on the underlying torus is again a contact map. Hence adding this extra dimension we still get the right dimension $2d - 4g + 4$ for a torus. Now that $\tau \geq 2g + 1$, the beginning degree of d is $8g + 1$ and any two consecutive such d differ by 3. We are done. $\square$.

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