1. Suppose that $X_1$ and $X_2$ form a random sample of size 2 from a distribution whose pdf is $f(x) = \frac{1}{2}x$ for $0 < x < 2$ and $f(x) = 0$ for other $x$. Find

(a) The distribution function $F$ of $X_1$.

(b) $Pr(\max(X_1, X_2) \leq \frac{3}{2})$.

(c) $Pr(\min(X_1, X_2) \leq \frac{3}{2})$.

(d) $Pr(\min(X_1, X_2) \geq 1$ and $\max(X_1, X_2) \leq \frac{3}{2})$.

2. Suppose that $X_1, X_2, X_3$ form a random sample from a uniform distribution on $[0, 1]$. Let

$$Y = X_1 + X_2 + X_3, \quad Z = \frac{1}{3}(X_1 + X_2 + X_3),$$

and let $g$ and $h$ denote the pdf's of $Y$ and $Z$, respectively.

(a) $g$ requires several different pieces for its specification. One piece is

$$g(x) = \frac{1}{2}(3-x)^2, \quad \text{if} \quad 2 < x < 3.$$

Verify this for yourself, and find the rest of $g$.

Suggestions: Denote the pdf of $X_1 + X_2$ by $g_2$. You can use the formula for $g_2$ we found in class. Note that $Y = (X_1 + X_2) + X_3$.

(b) Explain how to obtain $h$ from $g$. Then sketch, on the same set of axes, the graphs of $h$, the pdf of $\frac{1}{3}(X_1 + X_2)$, and the pdf of $X_1$. What do you think the pdf of $\frac{1}{n} \sum_{k=1}^{n} X_k$ will look like for large $n$ when $X_1, ..., X_n$ form a random sample of size $n$ from $U(0,1)$?
3. Let $X$ have the pdf $f(x) = \frac{1}{4}(1 + x)$ for $0 < x < 2$ and $f(x) = 0$ for other $x$. Define $Y_1 = 2X^4 - 1$ and $Y_2 = (X - 1)^4$.

(a) Find the ranges of $X$, $Y_1$ and $Y_2$.

(b) Is the function $r_1(x) = 2x^4 - 1$ one-one on the range of $X$? How about $r_2(x) = (x - 1)^4$?

(c) Calculate the pdf's $g_1$ and $g_2$ of $Y_1$ and $Y_2$.

In (c), you can use either the distribution function or the change of variable technique. I suggest the former. In (a) and (b) the term “range” means “set of values taken on by.” It does not agree with the term “range of a sample” defined on p.174.

4. Let $X$ be as in Problem 3.

(a) Find a formula for a function $r$ such that $Y = r(X)$ is uniformly distributed on $(0, 1)$.

(b) Find a formula for a function $q$ such that $W = q(X)$ has an exponential distribution with mean 1. This means that the pdf of $W$ equals $e^{-w}$ for $w > 0$ and equals zero for $w < 0$.

R1. Let $X$ and $Y$ be independent and discrete, with $X$ uniformly distributed on the integers $\{0, 1, 2, \}$ and $Y$ uniformly distributed on the integers $\{0, 1, 2, 3\}$. Use convolution to find the p.f. $h$ of $X + Y$.

Use the discrete convolution formula to find the pf of $X + Y$.

R2. Suppose that $X$ and $Y$ take on only positive values, and have joint pdf $f$. Let $g$ be the joint pdf of $U = X^2 - Y^2$ and $V = 2XY$. If $f(1, 1) = 5$, what is the value of $g(0, 2)$?
And what does the number $g(0, 2)$ signify in terms of the probability of something involving $U$ and $V$?

You may use the fact that the mapping $(x, y) \rightarrow (x^2 - y^2, 2xy)$ carries the first quadrant 1-1 onto the upper half space $y > 0$.

R3. Let $X$ and $Y$ have the joint pdf of HW4, Problem 5. Find the joint pdf $g$ of $U = X + Y$ and $V = X - Y$. Be sure to sketch the set on which $g \neq 0$.

R4. Let $X$ and $Y$ be independent random variables each of which takes on only the values 0 and 1. Suppose their joint p.f. satisfies $f(0, 0) = 1/2$ and $f(1, 0) = 1/4$. Find $f(0, 1)$, $f(1, 1)$, and the p.f.'s of $X$ and $Y$.

R5. Let $\Omega$ be the bounded region between the parabolas $y = x^2$ and $y = 2 - x^2$. Suppose that $(X, Y)$ is a random vector uniformly distributed on $\Omega$. Then $(X, Y)$ has a constant value on $\Omega$. Find

(a) The value of the constant.

(b) $Pr(Y < X)$.

(c) The marginal c.f. and marginal pdf of $Y$. The formulas require several different pieces. Check that your $F_2$ is continuous at the break points.

(d) Are $X$ and $Y$ independent? Justify your answer.

For §3.7, recommended problems are 1,3,4,6,7.