1. From a previous homework we have the following:

\[ f(x,y) = \begin{cases} \frac{2}{5} \frac{2}{2x+3y} & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ f_1(x) = \begin{cases} \frac{2}{5} (2x+3y) & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases} \quad f_2(y) = \begin{cases} \frac{2}{5} (3y+1) & 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases} \]

\[ g_1(x,y) = \frac{2x+3y}{1+3y} \quad 0 \leq x, y \leq 1 \]

\[ g_2(y|x) = \frac{2x+3y}{2x+3y} \quad 0 \leq x, y \leq 1 \]

a) Compute \( E(Y|X) \). Verify for yourself \( E(E(Y|X)) = E(Y) \)

\[ E(Y|X) = \int_0^1 \frac{2x+3y}{2x+3y} \, dy = \int_0^1 \frac{2xy+3y^2}{2x+3y} \, dy \]

\[ = \frac{1}{2x+\frac{3}{2}} \left[ (xy^2+y^3) \right]_{y=0}^{y=1} = \frac{x+1}{2x+\frac{3}{2}} \]

So \( E(Y|X) = \frac{x+1}{2x+\frac{3}{2}} \)

\[ E(E(Y|X)) = E\left( \frac{x+1}{2x+\frac{3}{2}} \right) = \int_0^1 \frac{2}{5} \frac{x+1}{2x+\frac{3}{2}} \, dx \]

\[ = \frac{2}{5} \int_0^1 (x+1) \, dx = \frac{2}{5} \left[ \left(\frac{x^2}{2} + x\right) \right]_0^1 = \frac{2}{5} \left( \frac{3}{2} \right) = \frac{3}{5} \]

\[ E(Y) = \int_0^1 \frac{2}{5} (3y^2+y) \, dy = \frac{2}{5} \left[ \left( y^3 + \frac{y^2}{2} \right) \right]_0^1 = \frac{2}{5} \left( \frac{3}{2} \right) = \frac{3}{5} \]

So \( E(E(Y|X)) = E(Y) \).
b) If the value of the math score $X$ is disregarded, what predicted value of $Y$ has the smallest m.s.e.? What is the value of this m.s.e.?

The predicted value of $Y$ with the smallest m.s.e. is $Y = E(Y) = 3/5$

$$\text{Var}(Y) = E(Y^2) - [E(Y)]^2$$

$$= \frac{2}{5} \int_0^1 (3y^3 + y^2) \, dy - \left( \frac{3}{5} \right)^2$$

$$= \frac{2}{5} \left[ \left( \frac{3}{4} y^4 + \frac{1}{2} y^3 \right) \bigg|_0^1 \right] - \frac{9}{25} = \frac{2}{5} \left( \frac{3}{4} + \frac{1}{2} \right) - \frac{9}{25}$$

$$= \frac{11}{150} \approx 0.073$$

So the value of this m.s.e. is $\text{Var}(Y) = \frac{11}{150} \approx 0.073$

c) If $X = 0.1$, what predicted $Y$ has the smallest m.s.e.? What is the value of this m.s.e.?

The predicted value of $Y$ with the smallest m.s.e. is $Y = E(Y | 0.1) = \frac{0.1 + 1}{2(0.1) + 1} = \frac{11}{17} \approx 0.647$

$$\text{Var}(Y | X) = E(Y^2 | X) - [E(Y | X)]^2$$

$$E(Y^2 | X) = \int_0^1 y^2 g_2(Y | X) \, dy = \frac{1}{2X + 3/2} \int_0^1 (2xy^2 + 3y^3) \, dy$$

So $\text{Var}(Y | 0.1) = \frac{10}{17} \int_0^1 (0.2y^2 + 3y^3) \, dy - \frac{121}{289} = \frac{107}{1734} \approx 0.0617$
If \( x=0.1, \) what predicted value of \( y \) has smallest m.a.e.? Write down a definite integral that gives the value of this m.a.e.

We want the median of \( g_2(y|0.1) \).

\[
\frac{1}{2} = \int_{0}^{\infty} \frac{2+3y}{2+3/2} \, dy \iff \frac{1}{2} = \frac{10}{17} \int_{0}^{\infty} (2+3y) \, dy
\]

\[
\iff \frac{17}{20} = 2m + \frac{3}{2} m^2
\]

\[
\iff 30m^2 + 4m - 17 = 0
\]

\[
\iff m = \frac{-4 + \sqrt{2086}}{100}
\]

So the predict value of \( y \) that has the smallest m.a.e. is \( y = \frac{-1}{15} + \frac{1}{30} \sqrt{514} \approx 0.689 \)

The value of this m.a.e. is \( E(|y-m|) = \int_{0}^{1} |y-m| g_2(y|0.1) \, dy = \int_{0}^{1} |y-m| \frac{2+3y}{1.7} \, dy \), where \( m \) is as above

Sketch a graph of \( E(y|x) \) over \( 0 \leq x \leq 1 \).

\[
E(y|x) = \frac{x+1}{2x+3/2}
\]

\[
E(y|0) = \frac{2}{3}
\]

\[
E(y|0.5) = 0.6
\]

\[
E(y|1) = \frac{4}{3}
\]
calculate \( \text{Cov}(X, Y) \) & \( \rho(X, Y) \).

From before we have \( \text{Var}(Y) = \frac{11}{150} \).

We compute the following:

\[
E(X) = \int_0^1 \left( 2x^2 + \frac{3}{2} x \right) \, dx = \frac{2}{5} \left[ \left( \frac{2}{5} x^3 + \frac{3}{4} x^2 \right) \right]_0^1 = \frac{2}{5} \left( \frac{2}{5} + \frac{3}{4} \right) = \frac{17}{60}
\]

\[
E(X^2) = \frac{2}{5} \int_0^1 (2x^3 + \frac{3}{2} x^2) \, dx = \frac{2}{5} \left( \frac{1}{2} x^4 + \frac{1}{2} x^3 \right) \bigg|_0^1 = \frac{3}{5}
\]

\[
\Rightarrow \text{Var} X = E(X^2) - E(X)^2 = \frac{3}{5} - \left( \frac{17}{60} \right)^2 = \frac{79}{1500}
\]

\[
E(XY) = \frac{2}{5} \int_0^1 \int_0^1 (2x^2y + 3y^2x) \, dy \, dx = \frac{2}{5} \int_0^1 (x^2 + x) \, dx = \frac{2}{5} \left( \frac{x^3}{3} + \frac{x^2}{2} \right) \bigg|_0^1 = \frac{1}{3}
\]

So \( \text{Cov}(X, Y) = E(XY) - \mu_X \mu_Y = \frac{1}{3} - \left( \frac{17}{60} \right) \left( \frac{3}{5} \right) = -\frac{1}{150} \)

Also, \( \rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = -\frac{\frac{1}{150}}{\sqrt{\left( \frac{11}{150} \right) \left( \frac{79}{1500} \right)}} \approx -0.0876 \)

These results indicate a negative association between students' math & music scores.

So an increase in math score is accompanied on the whole by a slight decrease in music score.
We are given $Q = s_1R_1 + s_2R_2$, $E(R_1) = 200$, $\text{Var}(R_1) = 4$, $\text{Var}(R_2) = 25$, $\rho(R_1, R_2) = -\frac{1}{2}$. Want to find $\text{Cov}(R_1, R_2)$ & a formula for $\text{Var}Q$.

\[
\rho(R_1, R_2) = \frac{\text{Cov}(R_1, R_2)}{\text{Var}R_1 \text{Var}R_2} \implies -\frac{1}{2} = \frac{\text{Cov}(R_1, R_2)}{(2)(5)}
\]

Thus, $\text{Cov}(R_1, R_2) = -5$

Now $\text{Var}Q = \text{Var}(s_1R_1 + s_2R_2) = s_1^2 \text{Var}(R_1) + s_2^2 \text{Var}(R_2) + 2s_1s_2 \text{Cov}(R_1, R_2)$.

So $\text{Var}Q = 4s_1^2 + 25s_2^2 - 10s_1s_2$

Suppose you want the mean return to be $1,000. So how many shares of stocks A & B should you buy so as to minimize the volatility of $Q$?

So we want $1000 = E(Q) = s_1E(R_1) + s_2E(R_2) = 100s_1 + 200s_2$

$\implies s_1 = 10 - 2s_2$

$\implies \text{Var}Q = 4(10 - 2s_2)^2 + 25s_2^2 - 10(10 - 2s_2)s_2$

So $(\text{Var}Q)' = -160(10 - 2s_2) + 50s_2 - 100 + 40s_2 = 18s_2 - 260$

The volatility of $Q$ is minimized when $(\text{Var}Q)' = 0$, which happens when $s_2 = \frac{130}{18}$

Note $s_2 = \frac{130}{18} \implies s_1 = 10 - 2(\frac{130}{18}) = \frac{350}{18}$
So the volatility of \( \text{Var}(q) \) is minimized when we buy
\[ s_1 = 35\% \] shares of A \\ & \[ s_2 = 15\% \] shares of B.

If we can only buy whole shares of stock, then the volatility of \( \text{Var}(q) \) is minimized when we buy
\[ s_1 = 6 \] shares of A \\ & \[ s_2 = 2 \] shares of B.

3] If a student is selected at random from the entire group that took the test, what is the expected value of her score?

Let \( X \) denote the number of her school (so \( X=1 \) for school A, \( X=2 \) for school B, & \( X=3 \) for school C).

Let \( Y \) denote her exam score.

\[
\text{Then } E(Y) = E[E(Y|X)] = (0.2)(80) + (0.3)(76) + (0.5)(84) = 80.2
\]