Instructions: Work 1, 2, 3 and any three of 4-7.

1. Suppose that $X_1, \ldots, X_n$ form a random sample from a distribution with pdf

$$f(x|\theta) = \theta x^{\theta-1}, \quad \text{for} \quad 0 < x < 1$$

and $f(x|\theta) = 0$ for other $x$, where $\theta$ is an unknown parameter with $0 < \theta < \infty$.

(a) Find a formula for the likelihood function $f_n(\overline{x}^n|\theta)$.

(b) Find a formula for the maximum likelihood estimator $\hat{\theta}_n$. Be sure to show how you found it.

(c) Calculate the maximum likelihood estimate when $n = 2$, $x_1 = 1/4$, and $x_2 = 1/2$.

2. Suppose that $X_1, \ldots, X_n$ form a random sample from a Poisson distribution with unknown mean $\lambda$.

(a) Find a formula for the likelihood function $f_n(\overline{x}^n|\lambda)$.

(b) Suppose that we assign as prior pdf of $\lambda$ the gamma distribution with parameters $\alpha$ and $\beta$. Then the posterior distribution of $\lambda$ is a *** distribution with parameters +++ and !!! . Fill in the the blanks, and write down a few equations or lines which show me you know how to derive this result.

(c) Suppose that $\alpha = 3$, $\beta = 5$, and that three observations give the values 4, 6 and 10. Find the Bayes estimate and the mean square error of the Bayes estimate using the squared error loss function.

3. Suppose that a point $(X_1, X_2, X_3)$ is to be chosen at random in 3- dimensional space, where $X_1, X_2, X_3$ are independent rv’s and each has a normal distribution with mean zero and variance 16. Let $Y = \sum_{i=1}^{3} X_i^2$. 
(a) Find a number $c$ such that $P_r(Y \leq c) = .95$.

(b) Find the mean and variance of $Y$.

4. If $X_1, \ldots, X_n$ is a random sample from a normal distribution with mean $\mu$ and variance $\sigma^2$, then $\overline{X}_n$ and $\hat{\sigma}^2$ are independent. Here $\hat{\sigma}^2$ is defined in Problem 6. To prove the independence when $\mu = 0$ and $\sigma = 1$ we expressed $\overline{X}_n$ and $\hat{\sigma}^2$ in terms of independent standard normals $Y_1, \ldots, Y_n$.

What were these expressions?

5. By definition, a random interval $I$ is a $\gamma$-confidence interval for a parameter $\theta$ if, when $\theta$ is the true parameter, the probability that *** equals !!. What are *** and !!?

6. Suppose that $X_1, \ldots, X_n$ form a random sample from a normal distribution for which both the mean $\mu$ and the variance $\sigma^2$ are unknown. The statistics

$$\overline{X}_n \quad \text{and} \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2$$

. can be used to construct confidence intervals for $\mu$. Find such a confidence interval with confidence coefficient 0.8 when $\overline{X}_n = 100$, $\hat{\sigma}^2 = 10$, and $n = 4$.

7. Let $z_1, \ldots, z_n$ and $\alpha$ be real numbers. There is an identity

$$\sum_{i=1}^{n} (z_i - \alpha \theta)^2 = A(\theta - B)^2 + K, \quad \forall \theta \in \mathbb{R},$$

where $A$, $B$ and $K$ do not depend on $\theta$.

Express $A$ and $B$ in terms of $n$, $\overline{z_n}$, and $\alpha$. 