Math 310 Solutions
Homework #2

1. Let $S = \{1\} \cup \{m \in \mathbb{N} : m \neq 1 \text{ and } m = q' \text{ for some } q \in \mathbb{N}\}$

First we'll show $S = \mathbb{N}$ using induction.

Well, $1 \in S$ by definition. Suppose $m \in S$. Then $m' \neq 1$ by PA #4, and m is a predecessor of m', so $m' \in S$. By induction, $S = \mathbb{N}$.

Now we'll show uniqueness. Let $m \in \mathbb{N}$, and suppose $m = q'$ and $m = r'$. Then $q' = r'$, so by PA #3, $q = r$. Thus, the predecessor exists and is unique.

2. i) Fix $m \in \mathbb{N}$. Let $S = \{n \in \mathbb{N} : mn = nm\}$.

First, since $m \cdot 1 = m$ and $1 \cdot m = m$, we have that $m \cdot 1 = 1 \cdot m$, so $1 \in S$.

Now assume $n \in S$. Then
$$\begin{aligned} mn' &= mn + m && \text{by definition 3} \\ &= nm + m && \text{by induction hypothesis} \\ &= n'm && \text{by Theorem 4} \end{aligned}$$

thus, $n' \in S$, so by induction, $S = \mathbb{N}$. Since m was arbitrary, we have that $mn = nm \ \forall m, n \in \mathbb{N}$.

ii) Fix $m, n \in \mathbb{N}$. Let $S = \{p \in \mathbb{N} : (mn)p = m(np)\}$.

First, since $(mn) \cdot 1 = mn = m(n \cdot 1)$, $1 \in S$.

Suppose $p \in S$. Then
$$\begin{aligned} (mn)p' &= (mn)p + mn && \text{by definition 3} \\ &= m(np) + mn && \text{by induction hypothesis} \\ &= m(np + n) && \text{by distribution} \\ &= m(np') && \text{by Theorem 4} \end{aligned}$$

Thus, $p' \in S$. By induction, $S = \mathbb{N}$. Since the choice of m, n was arbitrary, we may conclude that $(mn)p = m(np) \ \forall m, n, p \in \mathbb{N}$.

3. Suppose, in anticipation of a contradiction, that there exist $m, n \in \mathbb{N}$ such that $n < m$ and $m < n+1$.

Then there exist $p, q \in \mathbb{N}$ such that $n+p = m$ and $m+q = n+1$.

So
$$(n+p)+q = n+1 \quad \text{by substitution}$$
$$n+(p+q) = n+1 \quad \text{by associlivity}$$
$$p+q = 1 \quad \text{by cancellation}$$

But this contradicts HW #1, problem 3. Thus, no such m, n exist.

4. Let $S = \mathbb{N} \times \mathbb{N}$. Define an equivalence relation $\sim$ on S by setting $(x,y) \sim (u,v)$ when $xv(y+u) = yu(x+v)$.

i) First we'll show $\sim$ is actually an equivalence relation.

Reflexivity Since $xy(y+x) = yx(x+y)$ by commutivity of $+$ and $\cdot$, $(x,y) \sim (x,y)$.

Symmetry If $(x,y) \sim (u,v)$, we have $xv(y+u) = yu(x+v)$
$$\Rightarrow vx(u+y) = uy(v+x) \quad \text{by commutivity}$$
$$\Rightarrow uy(v+x) = vx(u+y)$$
$$\Rightarrow (u,v) \sim (x,y)$$

Transitivity Suppose $(x,y) \sim (u,v)$ and $(u,v) \sim (a,b)$.

Then $xv(y+u) = yu(x+v)$ and $ub(v+a) = va(u+b)$
$$\Rightarrow ab \cdot xv(y+u) = ab \cdot yu(x+v) \text{ and } xy \cdot ub(v+a) = xy \cdot va(u+b)$$

Distributing and adding the two equations together yields:
$$ab\cancel{xvy} + abxvu + xyubv + \cancel{xyub}a = ab\cancel{yux} + abyuv + xyvau + \cancel{xyvab}$$

Cancelling as above yields:
$$abxvu + xyubv = abyuv + xyvau$$
$$\Rightarrow uvxb(a+y) = uvya(b+x) \quad \text{commutivity, distributivity}$$
$$\Rightarrow xb(y+a) = ya(x+b) \quad \text{by cancellation, commutivity}$$
$$\Rightarrow (x,y) \sim (a,b)$$

ii) Let's determine which pairs (x,y) are equivalent to $(1,1)$.

$$\begin{aligned}
(x,y) \sim (1,1) &\Rightarrow x \cdot 1\,(y+1) = y \cdot 1\,(x+1) \\
&\Rightarrow x(y+1) = y(x+1) \\
&\Rightarrow xy + x = yx + y && \text{distribution} \\
&\Rightarrow xy + x = xy + y && \text{commutivity} \\
&\Rightarrow x = y && \text{cancellation}
\end{aligned}$$

Thus, any pair $(x,x) \sim (1,1)$, i.e.

$(2,2)$, $(3,3)$, $(4,4)$, etc.