1. Let $S = \{1\} \cup \{m \in \mathbb{N} : m+1 \text{ and } m = q^l \text{ for some } q \in \mathbb{N}\}$.

First we'll show $S = \mathbb{N}$ using induction.

Well, $1 \in S$ by definition. Suppose $m \in S$. Then $m+1$ by PA $\#4$, and $m$ is a predecessor of $m+1$ so $m \in S$. By induction, $S = \mathbb{N}$.

Now we'll show uniqueness. Let $m \in \mathbb{N}$, and suppose $m = q^l$ and $m = r^l$. Then $q^l = r^l$, so by PA $\#3$, $q = r$. Thus, the predecessor exists and is unique.

2. c) Fix $m \in \mathbb{N}$. Let $S = \{n \in \mathbb{N} : mn = nm\}$.

First, since $m \cdot 1 = m$ and $1 \cdot m = m$, we have that $m \cdot 1 = 1 \cdot m$, so $1 \in S$.

Now assume $n \in S$. Then $mn' = mn + m$ by definition 3

$= nm + m$ by induction hypothesis

$= n'm$ by Theorem 4

thus, $n' \in S$, so by induction, $S = \mathbb{N}$. Since $m$ was arbitrary, we have that $mn = nm \ L m, n \in \mathbb{N}$.

6) Fix $m, n \in \mathbb{N}$. Let $S = \{p \in \mathbb{N} : (mn)p = m(np)\}$.

First, since $(mn) \cdot 1 = mn = m(n \cdot 1)$, $1 \in S$.

Suppose $p \in S$. Then $(mn)p' = (mn)p + mn$ by definition 3

$= m(np) + mn$ by induction hypothesis

$= m(np + n)$ by distrib

$= m(np')$ by Theorem 4

Thus, $p' \in S$. By induction, $S = \mathbb{N}$. Since the choice of $mn$ was arbitrary, we may conclude that $(mn)p = m(np) \ L m, n, p \in \mathbb{N}$.
3. Suppose, in anticipation of a contradiction, that there exist \( n, m \in \mathbb{N} \) such that \( m < n \) and \( n < n+1 \).

Then there exist \( p, q \in \mathbb{N} \) such that \( n + p = m \) and \( n + q = n + 1 \).

So \( (n+p) + q = n + 1 \) by substitution

\[ n + (p+q) = n + 1 \] by associativity

\[ p+q = 1 \] by cancellation

But this contradicts HW #1, problem 3. Thus, no such \( m/n \) exist.

4. Let \( S = \mathbb{N} \times \mathbb{N} \). Define an equivalence relation \( \sim \) on \( S \) by setting \( (x, y) \sim (u, v) \) when \( xv(y+u) = yu(x+u) \).

i) First we'll show \( \sim \) is actually an equivalence relation.

**Reflexivity** Since \( xy(y+x) = yx(x+y) \) by commutativity of + and \( \cdot \)

\( (x, y) \sim (x, y) \).

**Symmetry** If \( (x, y) \sim (u, v) \), we have \( xy(y+u) = yu(x+u) \)

\[ \Rightarrow xv(y+u) = uy(x+u) \] by commutativity

\[ \Rightarrow uy(v+x) = vx(u+y) \]

\[ \Rightarrow (u, v) \sim (x, y) \)

**Transitivity** Suppose \( (x, y) \sim (u, v) \) and \( (u, v) \sim (a, b) \).

Then \( xv(y+u) = yu(x+v) \) and \( ub(v+a) = va(u+t) \)

\[ \Rightarrow ab \cdot xv(y+u) = ab \cdot yu(x+v) \] and \( xy \cdot ub(v+a) = xy \cdot va(u+t) \)

Distributing and adding the two equations together yields:

\[ abxy + abxu + xyuv + xyub = abyu + abuv + xyvau + xyvub \]

Cancelling as above yields:

\[ abxu + xyuv = abyu + xyvau \]

\[ \Rightarrow uvxb(a+y) = uxya(b+x) \] commutativity, distributivity

\[ \Rightarrow xb(y+a) = ya(x+b) \] by cancellation, commutativity

\[ \Rightarrow (x, y) \sim (a, b) \]
(ii) Let us determine which pairs \((x,y)\) are equivalent to \((1,1)\).

\[(x, y) = (1, 1) \Rightarrow x \cdot 1 = y \cdot 1 \Rightarrow x = y\]

Thus, any pair \((x, x) = (1,1)\), i.e.

\((2,2), (3,3), (4,4), \text{ etc.}\)