1) Find \( \lim_{x \to \infty} (xe^{\frac{1}{x}} - x) \)

Solution: \( \lim_{x \to \infty} (xe^{\frac{1}{x}} - x) = \lim_{x \to \infty} x(e^{\frac{1}{x}} - 1) \)

\[ \lim_{x \to \infty} x = \infty, \quad \lim_{x \to \infty} (e^{\frac{1}{x}} - 1) = e^0 - 1 = e^0 - 1 = 0 \]

This limit is a 0 \( \cdot \) \( \infty \) indeterminate form.

\[ \lim_{x \to \infty} (xe^{\frac{1}{x}} - x) \]

\[ = \lim_{x \to \infty} x(e^{\frac{1}{x}} - 1) = \lim_{x \to \infty} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} \left( \frac{0}{0} \right) \]

by L'Hospital

\[ \lim_{x \to \infty} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} \left( -\frac{1}{x^2} \right) \]

\[ = \lim_{x \to \infty} e^{\frac{1}{x}} = e^0 \]

\[ = e^0 = 1 \]

(over)
2) Find the point on the parabola $y = x^2$ that is closest to the point $(3,0)$.

Solution:

\[ l = \text{distance from } (x, y) \text{ to } (3, 0) \]

We want to minimize $l$, and equivalently we can minimize
\[ l^2 = (x-3)^2 + (y-0)^2. \]

Given $y = x^2$, minimize
\[ l^2 = (x-3)^2 + y^2. \]

\[ 2l \frac{dl}{dx} = 2(x-3) + 4x^3, \quad l \text{ is not zero} \]

\[ \frac{dl}{dx} = 0 \Rightarrow 2(x-3) + 4x^3 = 0 \Rightarrow 2x^3 + x - 3 = 0 \]

We know $x=1$ is a solution of $2x^3 + x - 3 = 0$.

By long division, $2x^3 + x - 3 = (x-1)(2x^2 + 2x + 3) = 0$.

Since $2x^2 + 2x + 3$ is always positive, so we only have one solution $x=1$.

\[ x=1 \]
\[ y = x^2 = 1 \]

\[ (1, 1) \text{ is the point on } y = x^2 \text{ which is closest to } (3, 0), \text{ and} \]
\[ l = \sqrt{(1-3)^2 + (1-0)^2} = \sqrt{4 + 1} = \sqrt{5}. \]