1) A particle is moving along the x-axis and its position for x ≥ 0 is given by the formula \( x = \frac{1}{3} t^3 - 2t^2 + 3t \). On what interval(s) is the velocity of the particle decreasing?

**Solution:** \( v = dx/dt = t^2 - 4t + 3 \), \( dv/dt = 2t - 4 = 2(t - 2) \). The velocity \( v \) is decreasing where \( dv/dt < 0 \). That is, on the open interval \((0, 2)\).

2) A rock is thrown vertically upward from the edge of a stand on the moon's surface, which is 10 feet above the surface. Its height in meters after \( t \) seconds is given by \( h(t) = 24t - 0.8t^2 + 10 \) (e.g. \( h(0) = 10 \)). Find the total distance traveled by the rock from the time it is thrown up until the time it passes the stand on its way down.

**Solution:** \( v = 24 - 1.6t = 0 \) when \( t = 24/1.6 = 15 \). That means that after 15 seconds the rock reaches its highest point. 15 seconds later it will pass the stand on the way down (you can check that \( h(30) = 10 \)). Total distance traveled will then be \( s(15) - s(0) + |s(30) - s(15)| = 180 + 180 = 360 \) meters.

3) Find an equation for the normal line to the curve \( y = x \tan(x) \) at the point \((\pi, 0)\).

**Solution:** \( dy/dx = \tan(x) + x \sec^2(x) \). For \( x = \pi \) we get \( dy/dx = \pi \) (\( \tan(\pi) = 0 \) and \( \sec(\pi) = -1 \)). Then slope of normal line is \(-\frac{1}{\pi}\) and equation of normal line is \( y = -\frac{1}{\pi}(x - \pi) = -\frac{1}{\pi}x + 1 \).

4) Eliminate the parameter to find a Cartesian equation for the curve \( x = -1 + 3 \sec(t) \quad y = 2 + 3 \tan(t) \)

**Solution:** \( x + 1 = 3 \sec(t) \) and \( y - 2 = 3 \tan(t) \). So \((x+1)^2 = 9 \sec^2(t)\) and \((y - 2)^2 = 9 \tan^2(t) \). From the identity \( 1 + \tan^2(t) = \sec^2(t) \) we get that \( 9 + 9 \tan^2(t) = 9 \sec^2(t) \).

So we get the cartesian equation \( 9 + (y - 2)^2 = (x+1)^2 \). This can also be written as \((x+1)^2 - (y - 2)^2 = 9\), which is a hyperbola.
5) From the parametric equations \( x = t - \sin(t), \ y = 1 - \cos(t) \), find the second derivative, \( \frac{d^2y}{dx^2} \), at \( t = \frac{\pi}{3} \).

**Solution:**

\[
y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 + \sin(t)}{1 - \cos(t)}.
\]

Next we have \( \frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} \).

Now \( dy'/dt = \frac{\cos(t)(1 - \cos(t)) - (1 + \sin(t))(\sin(t))}{(1 - \cos(t))^2} = \frac{\cos(t) - \sin(t) - 1}{(1 - \cos(t))^2} \)

and

\( dx/dt = (1 - \cos(t)) \).

Then \( \frac{dy'/dt}{dx/dt} = \frac{\cos(t) - \sin(t) - 1}{(1 - \cos(t))^2} = \frac{1 - \sqrt{3}}{2} = - (2\sqrt{3} + 2) \) at \( t = \frac{\pi}{3} \).

6) If \( f(x) = x \cdot \ln(e^{\sqrt{x}}) \), find \( f'(1) \).

**Solution:**

\( f(x) = x \cdot \sqrt{x} \cdot \ln(e) = x^{3/2} \). Then \( f'(x) = \frac{3}{2} \sqrt{x} \) and \( f'(1) = \frac{3}{2} \).

7) Find an equation for the tangent line to the curve \( x^3 + y^3 = 9xy \) at the point \( (2, 4) \).

**Solution:**

\( 3x^2 + 3y^2 \frac{dy}{dx} = 9y + 9x \frac{dy}{dx} \). For \( x = 2 \) and \( y = 4 \) we get

\( 12 + 48 \frac{dy}{dx} = 36 + 18 \frac{dy}{dx} \). So \( 30 \frac{dy}{dx} = 24 \) and \( \frac{dy}{dx} = \frac{4}{5} \).

Equation for tangent line is \( (y - 4) = \frac{4}{5} (x - 2) \).

8) If \( f(x) = (\tan^{-1}(x))^2 \) then \( f'(1) = \).

**Solution:**

\( f'(x) = 2 (\tan^{-1}(x)) \cdot \frac{1}{1 + x^2} \). Then \( f'(1) = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2} \).

9) Find the slope of the tangent line to the curve \( x \cdot \arctan(y) + x \cdot y = \frac{\pi + 4}{4} \) at the point \( (1, 1) \).

**Solution:**

\( \arctan(y) + \frac{x}{1 + y^2} \frac{dy}{dx} + y + x \frac{dy}{dx} = 0 \). For \( x = 1 \) and \( y = 1 \) we get

\( \frac{\pi}{4} + \frac{1}{2} \frac{dy}{dx} + 1 + \frac{dy}{dx} = 0 \). \( \frac{3}{2} \frac{dy}{dx} = - (\frac{\pi + 4}{4}) \). Then \( \frac{dy}{dx} = - (\frac{\pi + 4}{6}) \).
10) If \( f(x) = x \cdot \log_3(2^{\sqrt{x}}) \), find \( f'(1) \).

**Solution:**
\[
f'(x) = \log_3(2^{\sqrt{x}}) + x \cdot \frac{1}{\ln(3)} \cdot \frac{1}{2\sqrt{x}} \cdot \ln(2) 
\times 2^{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = 
\log_3(2^{\sqrt{x}}) + \frac{1}{2} \cdot \frac{\ln(2)}{\ln(3)} \sqrt{x}.
\]
\[f'(1) = \log_3(2) + \frac{1}{2} \cdot \frac{\ln(2)}{\ln(3)} = \frac{3}{2} \cdot \frac{\ln(2)}{\ln(3)}.\]

Another way of doing this: \( f(x) = x \cdot \frac{1}{\ln(3)} \ln(2^{\sqrt{x}}) = x \cdot \frac{1}{\ln(3)} \cdot \sqrt{x} \cdot \ln(2) = 
\frac{\ln(2)}{\ln(3)} \cdot x^{3/2}. \) So \( f'(x) = \frac{3}{2} \cdot \frac{\ln(2)}{\ln(3)} \cdot x^{1/2} \) and \( f'(1) = \frac{3}{2} \cdot \frac{\ln(2)}{\ln(3)} \) (as before).

11) If \( f(x) = x^{e^x} \) then find \( f'(1) \).

**Solution:** By logarithmic differentiation we have \( \ln(y) = \ln(x^{e^x}) = e^x \ln(x) \).
Then \( \frac{1}{y} \frac{dy}{dx} = e^x \ln(x) + e^x \frac{1}{x} \) and \( f'(x) = x^{e^x} (e^x \ln(x) + e^x \frac{1}{x}) \).
Finally we get that \( f'(1) = e \) (\( \ln(1) = 0 \) and \( 1^e = 1 \)).

12) If \( f(x) = \sin^{-1}(\tan(x)) \) then find \( f'(x) \).

**Solution:**
\[
f'(x) = \frac{1}{\sqrt{1 - \tan^2(x)}} \cdot \sec^2(x) = \frac{\sec^2(x)}{\sqrt{1 - \tan^2(x)}}.
\]

13) If \( f(x) = x \cdot 4^{-x^2} \) then find \( f'(x) \).

**Solution:**
\[
f'(x) = 4^{-x^2} + x \cdot \ln(4) 4^{-x^2} \cdot -2x = 4^{-x^2}(1 - 2 \ln(4) x^2).
\]

14) Use logarithmic differentiation to find \( \frac{dy}{dx} \) if \( y = \sqrt{\frac{x^3 + 1}{\tan(x) \cdot \sec(x)}} \).

**Solution:**
\[
\ln(y) = \frac{1}{4} \ln\left(\frac{x^3 + 1}{\tan(x) \cdot \sec(x)}\right) = \frac{1}{4} \left( \ln(x^3 + 1) - \ln(\tan(x)) - \ln(\sec(x)) \right).
\]
\[
\frac{1}{y} \frac{dy}{dx} = \frac{1}{4} \left( \frac{3x^2}{x^3 + 1} - \frac{\sec^2(x)}{\tan(x)} - \frac{\sec(x) \tan(x)}{\sec(x)} \right)
\]
Then \( \frac{dy}{dx} = \frac{1}{4} \cdot \sqrt{\frac{x^3 + 1}{\tan(x) \cdot \sec(x)}} \cdot \left( \frac{3x^2}{x^3 + 1} - \frac{\sec^2(x)}{\tan(x)} - \tan(x) \right) \).
15) For \( f(x) = 12 \log_8(\ln(x)) \), find \( f'(e) \).

**Solution:** \( f'(x) = \frac{12}{\ln(8)} \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x} \). Then \( f'(e) = \frac{12}{\ln(8) \cdot e} \).

16) There are two points where the curve \( x^2 + xy + y^2 = 9 \) crosses the \( x\)-axis. At those two points the **tangent lines** are parallel. Find the common **slope**.

( Hint: Point on the \( x\)-axis has coordinates \((a, 0)\)).

**Solution:** For \( y = 0 \) we get \( x^2 = 9 \). The points are then \((-3, 0), (3, 0)\).
By implicit differentiation we have \( 2x + y + x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0 \). For \( y = 0 \), \( 2x + x \frac{dy}{dx} = 0 \). For \( x = \pm 3 \) we have \( \frac{dy}{dx} = -2 \). Same slope.

17) Find \( \lim_{\theta \to 0} \cos(\frac{\pi \theta}{\sin(\theta)}) \). (Recall that \( \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1 \))

**Solution:** By limit laws for composites we have that
\( \lim_{\theta \to 0} (\frac{\pi \theta}{\sin(\theta)}) = \cos(\lim_{\theta \to 0}(\frac{\pi \theta}{\sin(\theta)})) = \cos(\pi \cdot \lim_{\theta \to 0}(\frac{\theta}{\sin(\theta)})) \).

Finally, \( \lim_{\theta \to 0}(\frac{\theta}{\sin(\theta)}) = \frac{1}{\lim_{\theta \to 0}(\frac{\sin(\theta)}{\theta})} = \frac{1}{1} = 1 \) and we get that
\( \lim_{\theta \to 0} \cos(\frac{\pi \theta}{\sin(\theta)}) = \cos(\pi) = -1 \).
18) At 2:00 PM sailboat B is 4 km south of sailboat A. After that A starts moving east at 4 km/hr and B starts moving east at 1 km/hr.
Find the rate of change of the distance between the two boats at 3:00 PM.

**Solution:** In this case \( t = 0 \) is 2:00 PM and we want the result at \( t = 1 \), 3:00 PM.

\[
\begin{array}{c|c|c|c}
 & A & \rightarrow & A \\
\hline
4 & s & \ast & \ast \\
\hline
B & y & \rightarrow & (x - y)
\end{array}
\]

In the upper diagram the letters on the left represent the positions of the two boats at \( t = 0 \), boat A above and boat B below. The two letters on the right represent the positions at a later time. The arrows are the directions that boats travel.
x is distance A traveled and y is the distance B traveled. (x and y vary in time).
We are given that always, \( \frac{dx}{dt} = 4 \) km/hr and \( \frac{dy}{dt} = 1 \) km/hr.
The problem is to find \( \frac{ds}{dt} \) when \( t = 1 \). The distance between them, s, is the hypotenuse of a right triangle with the other sides being 4 and \((x - y)\). So
\[
s^2 = 4^2 + (x - y)^2.
\]
When \( t = 1 \) we have \( x = 4 \), \( y = 1 \) giving us \( s = 5 \).
By implicit differentiation
\[
2s \frac{ds}{dt} = 2(x - y)(\frac{dx}{dt} - \frac{dy}{dt}).
\]
For \( t = 1 \) we have that \( 10 \frac{ds}{dt} = 6(4 - 1) \).
So \( \frac{ds}{dt} = \frac{9}{3} \) km/hr at 3:00 PM.

19) When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/min. At what rate is the plate's area increasing when the radius is 50 cm?

**Solution:** We have a circle of radius \( r \) and are given \( \frac{dr}{dt} = 0.01 \). The area \( A = \pi r^2 \),
so we have \( \frac{dA}{dt} = 2\pi r \frac{dr}{dt} \). When \( r = 50 \) cm \( \frac{dA}{dt} = 100 \pi (0.01) = \pi \) cm²/min.

20) The length of a rectangle is decreasing at the rate of 5 cm/sec while the width is increasing at the rate of 3 cm/sec. Find the rate of change of the diagonal when the length is 10 cm and the width is 15 cm. Is it increasing or decreasing?

**Solution:** If \( x = \) length and \( y = \) width then we are given \( \frac{dx}{dt} = -5 \), \( \frac{dy}{dt} = 3 \),
both represent cm/sec. If s is the diagonal it is the hypotenuse of a right triangle with the other sides x and y. So \( x^2 + y^2 = s^2 \) and by implicit differentiation
\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2s \frac{ds}{dt}.
\]
If \( x = 10 \) and \( y = 15 \) then \( s = \sqrt{325} = 5\sqrt{13} \).
For those values we get \( -100 + 90 = 10\sqrt{13} \frac{ds}{dt} \) and
\[
\frac{ds}{dt} = -\frac{1}{\sqrt{13}} \text{ cm/sec (e.g. it is decreasing).}
\]