1) Find the x and y coordinates of all points on the curve \( y = x^3 - 12x \) at which the tangent line is horizontal.
\( y' = 3x^2 - 12 = 0 \) for \( x = \pm 2 \). The points are \((-2, 16)\) and \((2, -16)\).

2) For \( r = (1 + \sec(\theta)) \cdot \sin(\theta) \), find \( \frac{dr}{d\theta} \).
\( r = \sin(\theta) + \tan(\theta) \) so \( \frac{dr}{d\theta} = \cos(\theta) + \sec^2(\theta) \).

3) For \( f(x) = \frac{e^x - e^{-x}}{x} \) find \( f'(1) \).
\( f'(x) = \frac{(e^x + e^{-x})x - (e^x - e^{-x})}{x^2} \) so \( f'(1) = \frac{2}{e} \).

4) Find an equation for the tangent line to the curve \( y = \left(\frac{\sin(x)}{1 + \cos(x)}\right)^2 \) at the point \( \left(\frac{\pi}{2}, 1\right) \).
\( \frac{dy}{dx} = 2\left(\frac{\sin(x)}{1 + \cos(x)}\right)\left(\frac{\cos(x)(1 + \cos(x)) + \sin^2(x)}{(1 + \cos(x))^2}\right) \). For \( x = \frac{\pi}{2}, \frac{dy}{dx} = 2 \).
So the tangent line is \( y - 1 = 2(x - \frac{\pi}{2}) \).

5) If \( f(\theta) = \ln\left(\frac{e^\theta}{1 + e^\theta}\right) \) then find \( f'(\theta) \).
\( \ln\left(\frac{e^\theta}{1 + e^\theta}\right) = \ln(e^\theta) - \ln(1 + e^\theta) = \theta - \ln(1 + e^\theta) \).
So \( f'(\theta) = 1 - \frac{e^\theta}{1 + e^\theta} = \frac{1}{1 + e^\theta} \).

6) Find an equation for the tangent line to the curve \( x^3 + y^3 - 9xy = 0 \) at the point \((2, 4)\).
3\(x^2 + 3y^2 y' - 9y - 9x y' = 0 \). If \( x = 2 \) and \( y = 4 \) we get
12 + 48 \( y' - 36 - 18 \) \( y' = 0 \) and \( y' = \frac{4}{5} \).

7) For the function \( f(x) = x^{2/3} \) find the y-coordinate of the absolute maximum point of the curve on the closed interval \(-2 \leq x \leq 3\).
\( y' = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}} \). It never equals 0 and it DNE at \( x = 0 \), so \( x = 0 \) is the only critical point. \( f(-2) = 1.5874, f(0) = 0 \) and \( f(3) = 2.08 \).
So \( y = 2.08 \) is the y-coordinate of the absolute maximum point.
8) Find the x-coordinates of all the points on the curve 
\[ y = x^3 + x^2 - 8x - 5 \] which are either local maximum or local minimum points. In each case state clearly which one they are.
\[ y' = 3x^2 + 2x - 8 = (3x - 4)(x + 2) = 0 \text{ when } x = -2 \text{ or } x = \frac{4}{3}. \]
Curve is inc. for \( x < -2 \) and \( x > \frac{4}{3} \) and it is decreasing \(-2 < x < \frac{4}{3}\).
So \( x = -2 \) is a local maximum and \( x = \frac{4}{3} \) is a local minimum.

9) Find the critical points (both the x and y coordinates) of
\[ f(x) = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}. \]
Then identify the intervals on which f is increasing and decreasing.
\[ f' = \frac{4}{3}x^{-2/3} - \frac{4}{3}x^{-1/3} = \frac{4x - 4}{3x^{1/3}}. \]
This equals 0 when \( x = 1 \) and DNE for \( x = 0 \).
(1, -3) and (0, 0) are the two critical points and the curve is decreasing for \( x < 1 \) and it is increasing for \( x > 1 \). (\( x^{1/3} \) is always nonnegative).

10) On the curve \( y = 4x^3 - x^4 \) find the intervals on which the curve is concave up and the intervals on which it is concave down.
\[ y' = 12x^2 - 4x^3, \quad y'' = 24x - 12x^2 = 12x(2 - x) = 0 \text{ when } x = 0 \text{ or } x = 2. \]
Concave up on \((0, 2)\) and concave down for \( x < 0 \) and \( x > 2 \).

11) Find the x and y coordinates of all the inflection points on the curve
\[ y = x^4 - 8x^3 + 18x^2. \]
\[ y' = 4x^3 - 24x^2 + 36x, \quad y'' = 12x^2 - 48x + 36 = 12(x - 1)(x - 3) = 0 \text{ for } x = 1 \text{ or } x = 3. \] Inflexion points are \((1, 11)\) and \((3, 27)\).

12) Find \[ \lim_{x \to 0} \frac{x - \sin(x)}{x^2} (0) = \lim_{x \to 0} \frac{1 - \cos(x)}{3x^2} (0) = \lim_{x \to 0} \frac{\sin(x)}{6x} (0) = \]
\[ \lim_{x \to 0} \frac{\cos(x)}{6} = \frac{1}{6}. \]

13) For the limit \[ \lim_{x \to -\infty} (\ln(x))^{1/x} \] first state which type of indeterminate form it is and then find the limit.
It is the \( \infty^0 \) type, so we look at \[ \ln[ (\ln(x))^{1/x}] = \frac{1}{x} \ln[\ln(x)] = \frac{\ln[\ln(x)]}{x}. \]
Now we calculate \[ \lim_{x \to -\infty} \frac{\ln[\ln(x)]}{x} (\infty) = \lim_{x \to -\infty} \frac{1}{\ln(x)} \frac{1}{x} = 0. \] Hence
\[ \lim_{x \to -\infty} (\ln(x))^{1/x} = e^0 = 1. \]
14) Estimate the area under the curve \( y = \frac{1}{x} \) between \( x = 1 \) and \( x = 7 \) using \( M_3 \).
\[
\Delta x = 2 \text{ and midpoints are } x = 2, 4, 6. \text{ So } M_3 = \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{6} \right) 2 = 1.8333.
\]

15) Evaluate \( \int_{-1}^{4} |x - 2| \, dx = \int_{-1}^{2} -(x - 2) \, dx + \int_{2}^{4} (x - 2) \, dx = (2x - \frac{x^2}{2}) \bigg|_{-1}^{2} + (\frac{x^2}{2} - 2x) \bigg|_{2}^{4} = \frac{9}{2} + 2 = \frac{13}{2}. \)

16) Find \( \frac{dy}{dx} \) if \( y = \int_{\tan(x)}^{0} \frac{1}{1+t^2} \, dt = -\int_{0}^{\tan(x)} \frac{1}{1+t^2} \, dt. \) By the FundThm and the chain rule \( \frac{dy}{dx} = -\left( \frac{1}{1+\tan^2(x)} \right)(\sec^2(x)). \)

17) Evaluate \( \int_{-1}^{0} \left( \frac{\sqrt{1} - 1}{\sqrt{t}} \right)^2 \, dt. \) Let \( u = \sqrt{t} - 1, \ du = \frac{1}{2\sqrt{t}} \, dt. \)
\[
\text{Then } \int_{-1}^{0} \left( \frac{\sqrt{1} - 1}{\sqrt{t}} \right)^2 \, dt = 2 \int_{0}^{1} u^3 \, du = \frac{u^4}{2} \bigg|_{0}^{1} = 8.
\]

18) Using the substitution \( u = x^2 + 1 \) solve the indefinite integral
\[
\int x^3 \sqrt{x^2 + 1} \, dx = \int x^2 \cdot x \sqrt{x^2 + 1} \, dx
\]
\[
\text{du} = 2x \, dx, \ x = \frac{1}{2u} \, du, \ u^2 = u + 1.
\]
\[
\int x^2 \sqrt{x^2 + 1} \, dx = \frac{1}{2} \int (u - 1) \sqrt{u} \, du = \frac{1}{2} \int u^{\frac{3}{2}} - u^\frac{1}{2} \, du = \frac{1}{2} \left( \frac{2}{5} u^{\frac{3}{2}} - \frac{3}{4} u^{\frac{3}{2}} \right) + C = \frac{1}{5} (x^2 + 1)^{\frac{5}{2}} - \frac{1}{3} (x^2 + 1)^{\frac{3}{2}} + C
\]

19) Find the area of the region enclosed by the parabolas \( x = 8 - y^2 \)
and \( y = y^2 \). Intersection points are \( (4, 2) \) and \( (4, -2) \).
\[
\text{Area} = \int_{y=-2}^{y=2} (8 - y^2) \, dy = \int_{y=-2}^{y=2} 8 - 2y^2 \, dy = 2 \int_{0}^{2} 8 - 2y^2 \, dy = 2(8y - \frac{2}{3}y^3) \bigg|_{0}^{2} = 2 \left( 16 - \frac{16}{3} \right) = \frac{64}{3}.
\]

Another possibility is integral in \( x \)-coordinate. We will need two integrals.
\[
\text{Area} = \int_{x=0}^{x=4} \sqrt{x} - \left(-\sqrt{x}\right) \, dx + \int_{x=4}^{x=8} \sqrt{8-x} - \left(-\sqrt{8-x}\right) \, dx = \int_{x=0}^{x=4} 2 \sqrt{x} \, dx + \int_{x=4}^{x=8} 2 \sqrt{8-x} \, dx = \frac{4}{3} x^{\frac{3}{2}} \bigg|_{0}^{4} + \left( -\frac{4}{3} (8-x)^{\frac{3}{2}} \right) \bigg|_{4}^{8} = \frac{32}{3} + \left[ 0 - \left( -\frac{32}{3} \right) \right] = \frac{64}{3}.
\]
20) Find the area of the region enclosed by the curve \( y = x^3 \) and the line \( y = x \).

(Hint: In this problem the top and bottom curves change position in the middle).

Intersection points are \((-1, -1), (0, 0), (1, 1)\).

For \(-1 \leq x \leq 0\) \(y = x^3\) is top and \(y = x\) is the bottom and for \(0 \leq x \leq 1\) it reverses, \(y = x\) is top and \(y = x^3\) is bottom. Hence

\[
\text{Area} = \int_{-1}^{0} x^3 - x \, dx + \int_{0}^{1} x - x^3 \, dx = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.
\]

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Problem done in class:

Which definite integral is defined by the following Riemann sum:

\[
\lim_{n \to \infty} \sum_{k=1}^{n} \sqrt{2 + \frac{k}{n}} \left( \frac{2}{n} \right).
\]

If you are given a number of choices, \(\int_{a}^{b} f(x) \, dx\), here is what to look for.

Since in the above Riemann sum \(\Delta x = \frac{2}{n}\) we know that \(b - a\) must be equal to 2. Also need to check the integrand, \(f(x)\).

The first term in the summand is \(\sqrt{2 + \frac{2}{n}}\) \((k = 1)\) so we need that \(f(a + \frac{2}{n}) = \sqrt{2 + \frac{2}{n}}\), similarly \(f(a + \frac{4}{n}) = \sqrt{2 + \frac{4}{n}}\) \((k = 2)\), etc.

For example \(\int_{0}^{2} \sqrt{2 + x} \, dx\) works since \(2 - 0 = 2\). Also with \(a = 0\) and \(f(x) = \sqrt{2 + x}\), \(f(a + \frac{2}{n}) = f(0 + \frac{2}{n}) = f(\frac{2}{n}) = \sqrt{2 + \frac{2}{n}}\), OK.

Similarly \(\int_{1}^{3} \sqrt{1 + x} \, dx\) works. First \(3 - 1 = 2\). Then with \(a = 1\), \(f(x) = \sqrt{1 + x}, f(a + \frac{2}{n}) = f(1 + \frac{2}{n}) = \sqrt{1 + (1 + \frac{2}{n})} = \sqrt{2 + \frac{2}{n}}\), OK.

If the integral were \(\int_{1}^{4} f(x) \, dx\), for any \(f(x)\), since \(4 - 1 = 3\), not OK.

Also \(\int_{1}^{3} \sqrt{2 + x} \, dx\) is no good since \(f(x) = \sqrt{2 + x}\), then \(f(a + \frac{2}{n}) = f(1 + \frac{2}{n}) = \sqrt{2 + (1 + \frac{2}{n})} = \sqrt{3 + \frac{2}{n}}\) which is not \(\sqrt{2 + \frac{2}{n}}\) (not OK).