There are 18 questions on this exam, each question is worth 5 points. Your grade will be calculated by taking your score on the exam multiplied by $\frac{10}{9}$. In each question find the correct answer and show how you got it. A correct answer with no explanation might not get any points. The same thing might apply to an answer you get just from using your calculator.

1) Find $\lim_{x \to 1} \frac{e^{x-1} - 1}{2x-2} = \lim_{x \to 1} \frac{e^{x-1}}{2} = \frac{e^0}{2} = \frac{1}{2}$

2) For $f(x) = \frac{e^x}{\sin(x)}$ on the open interval $0 < x < \pi$, find on which part $f(x)$ is increasing.

$$f'(x) = \frac{e^x \sin(x) - e^x \cos(x)}{\sin^2(x)} = \frac{e^x \left( \sin(x) \cos(x) \right)}{\sin^2(x)}$$

$$f'(x) = 0 \quad \text{if} \quad \sin(x) = \cos(x) \quad 0 < x < \pi$$

$$\Rightarrow \quad x = \frac{\pi}{4}$$

$$f' \left( \frac{\pi}{4} \right) < 0 \quad \text{and} \quad f' \left( \frac{\pi}{4} \right) > 0$$

$f$ increasing for $\frac{\pi}{4} < x < \pi$
3) Find all the critical numbers of \( f(x) = \sqrt[3]{x^2 - 8x} \)

\[
f'(x) = \frac{1}{3} \left( x^2 - 8x \right)^{-2/3} (2x - 8) = \frac{2x - 8}{3 (x^2 - 8x)^{2/3}}
\]

\[
f' = 0 \quad \text{if} \quad x = 4
\]

\[
f' \text{ DNE if } x = 0, 8 \quad \text{(denominator is zero)}
\]

All critical numbers are \( x = 0, 4, 8 \)

4) Find the absolute maximum and absolute minimum values of

\( f(x) = 2x^3 - 6x + 1 \) on the closed interval \([-2, 2]\).

(i.e. the largest and smallest values the function takes on in that interval.)

\[
f'(x) = 6x^2 - 6 = 6(x^2 - 1) = 6(x + 1)(x - 1) = 0
\]

\( x = -1, 1 \) are all critical points

\[
f(-2) = -3, \quad f(-1) = 5, \quad f(1) = -3, \quad f(2) = 5
\]

\[
\text{abs max is 5} \quad \text{abs min is -3}
\]
5) Consider the function \( f(x) = 5x^4 - 3x^2 \). Find the antiderivative \( F(x) \) of \( f(x) \) with initial condition \( F(2) = 10 \).

\[
F(x) = x^5 - x^3 + C
\]

Given: \( F(2) = 32 - 8 + C \)

So \( C = -14 \)

\[
F(x) = x^5 - x^3 - 14
\]

6) Find \( \lim_{x \to \infty} \frac{\ln(4x+4)}{\ln(8x+2)} \)

\[
L.H.s. = \lim_{x \to \infty} \frac{\ln(4x+4)}{\ln(8x+2)} = \frac{\frac{4}{4x+4}}{\frac{8}{8x+2}} = \frac{4x+2}{8x+2} = \frac{1}{2}
\]

\[
L.H.s. = \lim_{x \to \infty} \frac{\ln(4x+4)}{\ln(8x+2)} = \frac{4x+1}{4x+2} = 1
\]
7) Find \( \lim_{x \to 1^-} (x)^{\frac{1}{x}} \).

Then, let

\[
y = \lim_{x \to 1^-} x^{\frac{1}{x-1}}
\]

\[
\ln(y) = \lim_{x \to 1^-} \ln(x^{\frac{1}{x-1}})
\]

\[
= \lim_{x \to 1^-} \frac{\ln(x)}{x-1} \quad \left( \frac{0}{0} \right)
\]

\[
= \lim_{x \to 1^-} \frac{-1}{x} = 1
\]

Then

\[
\ln(y) = 1
\]

\[
y = e^1
\]

\[
\Rightarrow \lim_{x \to 1^-} x^{\frac{1}{x-1}} = e
\]
8) Find the x coordinates of all the **inflection** points of the curve $f(x) = 3x^5 - 5x^3 + 3$.

$$f'(x) = 15x^4 - 15x^2$$
$$f''(x) = 60x^3 - 30x$$
$$= 30x(2x^2 - 1) = 0$$

if $x = 0$ or $x = \pm \frac{1}{\sqrt{2}}$

$$2x^2 - 1 = 0 \Rightarrow 2x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$$

| $f''$ | $-\frac{1}{\sqrt{2}}$ | $+$ | $0$ | $-$ | $+$ |

**Inflection points at**

$$x = -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$$
9) Among all cylinders whose surface area is 200 cm² find the radius of that one which has the largest possible volume of those types of cylinders. (surface area is \(2\pi rh + 2\pi r^2\), where \(r = \text{radius} \) and \(h = \text{height}\)).

\[
\text{Given: } 2\pi rh + 2\pi r^2 = 200
\]
\[
2\pi rh = 200 - 2\pi r^2
\]
\[
h = \frac{200 - 2\pi r^2}{2\pi r}
\]

We want to maximize the volume.

\[
V = \pi r^2 h = \pi r^2 \left(\frac{200 - 2\pi r^2}{2\pi r}\right)
\]

Or
\[
V(r) = 100r - \pi r^3
\]

\[
V'(r) = 100 - 3\pi r^2 = 0
\]

If
\[
3\pi r^2 = 100
\]
\[
r^2 = \frac{100}{3\pi}
\]

\[
r = \sqrt{\frac{100}{3\pi}}
\]

\[
r = \frac{10}{\sqrt{3\pi}}
\]
10) A street light is at the top of a 15 foot pole and a child 3 feet tall walks away from the pole at speed of 4 feet per second. How fast is the tip of the child’s shadow moving when the child is 20 feet from the pole?

\[ x = \text{distance of child from pole} \]
\[ y = \text{distance of tip of shadow from pole} \]

\[ \frac{y}{15} = \frac{y-x}{3} \]

By similar triangles:

\[ 3y = 15y - 15x \]
\[ 12y = 15x \]

Then

\[ \frac{dy}{dt} = 15 \frac{dx}{dt} \]

\[ \frac{dy}{dt} = \frac{5}{9} \frac{dx}{dt} = \left( \frac{5}{9} \right) (4) = 5 \frac{ft}{s} \]
11) \( f(x) \) is a curve with domain \( 0 \leq x \leq \pi \). If the derivative of \( f(x) \) is given by the formula \( f'(x) = e^x \sin(x) \), then find where the graph of \( f(x) \) is **concave up** and where it is **concave down**?

(We want information about the graph of \( f(x) \), not the graph of \( f'(x) \))

\[
f''(x) = e^x \sin(x) + e^x \cos(x) \\
= e^x (\sin(x) + \cos(x)) = 0
\]

if \( \sin(x) = -\cos(x) \), \( 0 \leq x \leq \pi \).

Only place is \( x = \frac{3\pi}{4} \).

\[
\begin{array}{cccc}
0 & + & 1 & - \\
& & \frac{3\pi}{4} & \pi \\
\end{array}
\]

**Concave up:** \( 0 \leq x < \frac{3\pi}{4} \)

**Concave down:** \( \frac{3\pi}{4} < x < \pi \)
12) Find all the vertical and horizontal asymptotes of the function

\[
f(x) = \frac{x^2 - 9}{2x^2 + 2x - 12}.
\]

**Horizontal:**
\[
\lim_{x \to \infty} \frac{x^2 - 9}{2x^2 + 2x - 12} = \lim_{x \to \infty} \frac{x^2 \left(1 - \frac{9}{x^2}\right)}{2x \left(x - \frac{12}{x^2}\right)} = \frac{1}{2}.
\]
So \(y = \frac{1}{2}\) is horizontal asymptote.

**Vertical:**
\[
2x^2 + 2x - 12 = 2(x^2 + x - 6) = 2(x+3)(x-2) = 0 \implies x = 2 \text{ or } x = -3.
\]
So
\[
\lim_{x \to -3} \frac{x^2 - 9}{2x^2 + 2x - 12} = 0, \quad \frac{\text{Loops } 0}{\text{both } x=2, x=-3 \text{ are vertical asymptotes}}
\]
but
\[
\lim_{x \to 2} \frac{x^2 - 9}{2x^2 + 2x - 12} = \lim_{x \to 2} \frac{(x+3)(x-3)}{2(x+3)(x-2)} = -6/10 = 3/5 \neq \infty.
\]
Hence, only vertical asymptote is \(x = 2\).
13) Find the **absolute minimum value** (i.e., smallest value of \( f(x) \)) for the function \( f(x) = x^3 - 12x + 1 \) on the closed interval \([-3, 5]\).

\[
\frac{df}{dx} = 3x^2 - 12 = 3(x^2 - 4) = 3(x - 2)(x + 2) = 0
\]

*Critical points are \( x = \pm 2 \)

\( f(-3) = 17 \quad f(-2) = 17 \quad f(2) = -15 \quad f(5) = 66 \)

Hence: \[ \text{absolute Min. } f(2) = -15 \]

14) Estimate the area under the graph of the function \( f(x) = e^{-x} \) over the interval \([-1, 1]\), using **four approximating rectangles** and left endpoints.

\[
\Delta x = \frac{1 - (-1)}{4} = \frac{1}{2}
\]

\[
L_R = \left[ f(-1) + f(-0.5) + f(0) + f(0.5) \right] \cdot \frac{1}{2}
\]

\[
L_R = \left( e^{-1} + e^{-0.5} + e^{0} + e^{-0.5} \right) \cdot 0.5
\]

\[
\approx 2.986767
\]
15) Find the largest area that a rectangle can have if it is inscribed in the upper semi-circle \( x^2 + y^2 = 1, \ y \geq 0 \). (i.e., the rectangle must have its base on the x-axis and its upper two corners on the circle \( x^2 + y^2 = 1 \).)

\[ y = \sqrt{1-x^2} \]

Area is \( A = (2x)(y) = 2x\sqrt{1-x^2} \)

\[ A'(x) = 2\sqrt{1-x^2} + (2x) \frac{(-2x)}{2x\sqrt{1-x^2}} \]

\[ = \frac{a(1-x^2) - 2x^2}{\sqrt{1-x^2}} = 0 \]

\[ 4x^2 = 2 \text{ or } x = \frac{1}{\sqrt{2}} \]

then \( y = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}} \)

and \( \frac{\text{MAX}}{\text{Area}} = 2 \left( \frac{1}{\sqrt{2}} \right) \left( \frac{1}{\sqrt{2}} \right) = 1 \)
16) Sketch a graph of the object in the \( x-y \) coordinate system whose area is described by the definite integral \( \int_{-1}^{2} (3 + 2x) \, dx \). That means draw the four straight lines which are the boundaries of that object.

![Graph of a function with boundaries](image)

17) Find the indefinite integral (i.e. general antiderivative) of the function 
\[
f(x) = \sqrt[3]{x^3} + \sqrt[4]{x^4} = x^{3/4} + x^{4/3}
\]

\[
\int x^{3/4} + x^{4/3} \, dx = \frac{4}{7} x^{7/4} + \frac{3}{7} x^{7/3} + C
\]

\[
F(x) = \frac{4}{7} x^{7/4} + \frac{3}{7} x^{7/3} + C
\]

General antiderivative
18) Find the limit  \( \lim_{x \to \infty} (x \cdot e^{1/x} - x) \). (Hint: factor out \( x \) to get a product, then rewrite it as an indeterminate form \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \) and use L'Hospital's rule.)

\[
= \lim_{x \to \infty} x (e^{1/x} - 1) (\infty, 0)
\]

\[
= \lim_{x \to \infty} \frac{e^{1/x} - 1}{\frac{1}{x}} (\frac{0}{0})
\]

\[
= \lim_{x \to \infty} \frac{e^{1/x} \cdot \frac{-1}{x^2}}{-\frac{1}{x}}
\]

\[
= \lim_{x \to \infty} e^{1/x} = e^0 = 1
\]