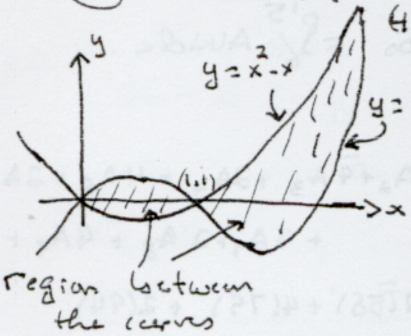
HW # 4 SOLUTIONS

MATH 1323

① [#26, p.453, §6.1]



The two curves $y = x^2 - x$ and $y = x^3 - 4x^2 + 3x$ intersect when

$$0 = (x^2-x) - (x^3-4x^2+3x)$$
$$= -(x^3 - 5x^2 + 4x)$$
$$= -x(x-1)(x-4)$$

i.e. for $x = 0, 1,$ or 4

$$\text{Area of region} = \int_0^4 \left|(x^2-x) - (x^3-4x^2+3x)\right| dx$$

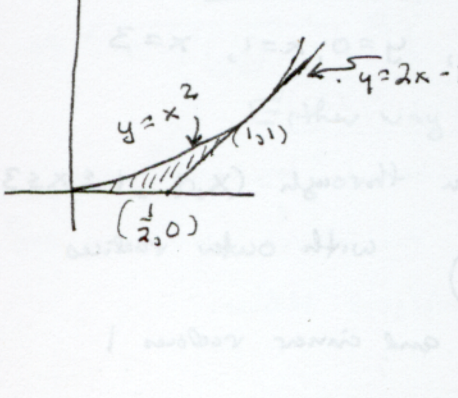
$$= \int_0^1 (x^3-5x^2+4x) - \int_1^4 (x^3-5x^2+4x)\,dx$$

$$= \left(\frac{x^4}{4} - \frac{5x^3}{3} + 2x^2\right)\Big|_0^1 - \left(x^4 - \frac{5x^3}{3} + 2x^2\right)\Big|_1^4$$

$$= 2\left(1/4 - 5/3 + 2\right) - \left(4^4 - \frac{5(4^3)}{3} + 2(4^2)\right)$$

$$= \boxed{11\,5/6 \text{ or } 11.8\overline{3}}$$

② [#36, p.453, §6.1] Find the area bounded by the parabola $y=x^2$, the tangent line to the parabola at $(1,1)$, and the x-axis. Since $\frac{d}{dx}(x^2) = 2x = 2$ at $x=1$, the tangent line has the equation $y-1 = 2(x-1)$ or $y = 2x-1$ with x-intercept at $(1/2, 0)$



The area bounded by the 3 curves is

$$\int_0^{1/2} (x^2-0)\,dx + \int_{1/2}^1 \underbrace{\left(x^2 - (2x-1)\right)}_{(x-1)^2} dx$$

$$= x^3/3 \Big|_0^{1/2} + \frac{(x-1)^3}{3}\Big|_{1/2}^1$$

$$= \left(\frac{1}{8}\right)\frac{1}{3} + 0 + \left(\frac{1}{8}\right)\left(\frac{1}{3}\right) = \boxed{\frac{1}{12}}$$

③ [# 19, p.464, §6.2]

The volume we're estimating is $\int_0^{15} A(x)\,dx$.

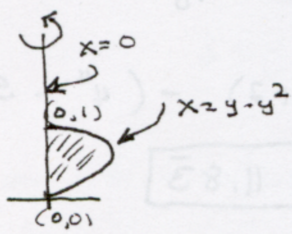
$$S_{10} = \tfrac{2}{3} M_5 + \tfrac{1}{3} T_5$$

$$= \tfrac{1}{3}\left(\tfrac{15-0}{10}\right)\left[A_0 + 4A_1 + 2A_2 + 4A_3 + 2A_4 + 4A_5 + 2A_6 + 4A_7 + 2A_8 + 4A_9 + A_{10}\right]$$

$$= \tfrac{1}{3}(1.5)\left[0 + 4(18) + 2(58) + 4(79) + 2(94) + 4(106) + 2(117) + 4(128) + 2(63) + 4(39) + 0\right]$$

$$= \boxed{1072 \text{ cm}^3}$$

④ [# 8, p.464, §6.2] Rotate the region bounded by $x = y - y^2$ and $x = 0$ about the y-axis and find the volume



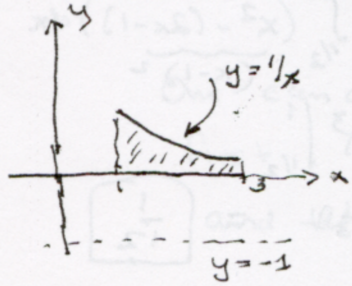
The cross-sectional disk at height y $(0 \le y \le 1)$ has radius $x = y - y^2$

$$\text{Volume} = \pi \int_0^1 (y - y^2)^2\,dy = \pi \int_0^1 (y^2 - 2y^3 + y^4)\,dy$$

$$= \pi\left(\tfrac{1}{3} - \tfrac{2}{4} + \tfrac{1}{5}\right)$$

$$= \boxed{\pi/30 = .105}$$

⑤ [# 10, p.463, §6.2] Rotate about $y = -1$ the region bounded by $y = 1/x$, $y = 0$, $x = 1$, $x = 3$ and find the volume.



A typical cross section through $(x,0)$, $1 \le x \le 3$ is a washer

with outer radius $1/x - (-1) = \frac{1}{x} + 1$ and inner radius 1

⑤ Continued. The volume is

$$V = \pi \int_1^3 \left\{ \left(\tfrac{1}{x}+1\right)^2 - 1^2 \right\} dx$$

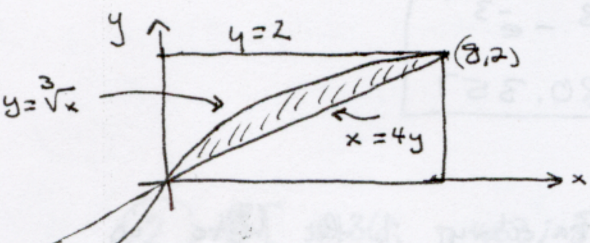
$$= \pi \int_1^3 \left(\frac{2}{x} + \frac{1}{x^2} \right) dx$$

$$= \pi \left(2\ln x - 1/x \right) \Big|_1^3$$

$$= 2\pi \ln 3 + 2\pi\left(1/3\right)$$

$$= \boxed{8.997}$$

⑥ [#14, p.463, §6.2] Find the volume of the solid obtained by rotating the $\overbrace{\text{region}}^{\text{in the 1st quadrant}}$ bounded by $x = 4y$ and $y = \sqrt[3]{x}$ about the line $y = 2$



As indicated in the sketch, the intersection points in the first quadrant are at (0,0) and (8,2)

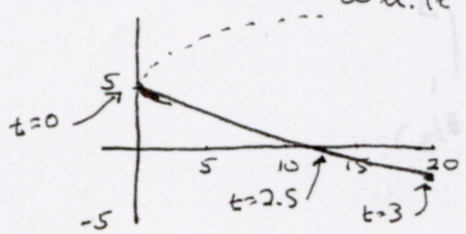
Disk Method: $$V = \pi \int_{x=0}^{8} \left\{ \underbrace{\left(2 - x/4\right)^2}_{\text{outer radius}} - \underbrace{\left(2 - \sqrt[3]{x}\right)^2}_{\text{inner radius}} \right\} dx$$

Shell Method: $$V = 2\pi \int_{y=0}^{2} \underbrace{(2-y)}_{\text{radius of shell}} \underbrace{(4y - y^3)}_{\text{length of shell}} \, dy$$

Either way, $\boxed{V = 23.457}$

(7) [# 6, p. 471, § 6.3] Sketch and find the arc length of the curve C parametrized by $\left.\begin{array}{l} x = e^t + e^{-t} \\ y = 5 - 2t \end{array}\right\} \; 0 \le t \le 3$

For $0 \le t \le 3$, $5 - 2t$ decreases from 5 to -1 while $x = e^t + e^{-t}$ increases from 2 to $e^3 + e^{-3} = 20.135$



$$L(C) = \int_0^3 \sqrt{\left(\frac{d}{dt}(e^t + e^{-t})\right)^2 + \left(\frac{d}{dt}(5-2t)\right)^2}\, dt$$

$$= \int_0^3 \sqrt{(e^t - e^{-t})^2 + 4}\;\; dt$$

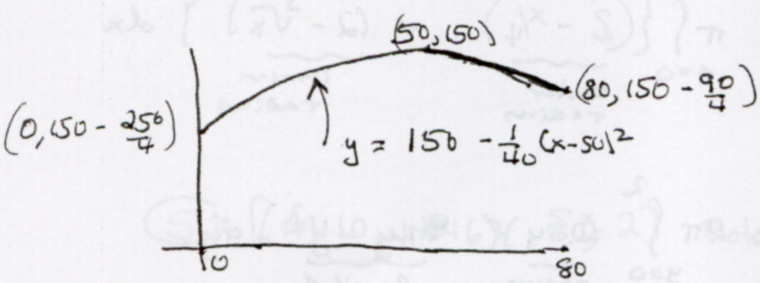
$$= \int_0^3 \sqrt{(e^t + e^{-t})^2}\; dt$$

$$= \int_0^3 (e^t + e^{-t})\, dt$$

$$= (e^t - e^{-t})\Big|_0^3$$

$$\boxed{\begin{aligned} &= e^3 - e^{-3} \\ &= 20.357 \end{aligned}}$$

(8) [# 22, p. 471, § 6.3] The trajectory of the kite is given by $y = 150 - \frac{1}{40}(x-50)^2$ for $0 \le x \le 80$



Distance traveled by the kite

= arc length of the trajectory

$$= \int_0^{80} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^{80} \sqrt{1 + \left(\frac{1}{20}\right)^2 (x-50)^2}\, dx$$

(by TI-83) $= \boxed{122.776}$

One can also grind out the integral "by hand" using first $\frac{x-50}{20} = \tan\theta$ to get an integral involving $\sec^3\theta$ and then using trig identities or integral tables.

MATH 1323

HOMEWORK #5 SOLUTIONS

① [#10, p.475, §6.4]

We wish to find b for which $3 = \frac{1}{b}\int_0^b (2+6x-3x^2)\,dx$

$$= \frac{1}{b}\left(2x+3x^2-x^3\right)\Big|_0^b$$

$$= 2+3b-b^2$$

Then $\quad b^2-3b+1 = 0$

$$b = 3/2 \pm \tfrac{1}{2}\sqrt{9-4}$$

$$= \frac{3\pm\sqrt{5}}{2}$$

$$= \boxed{2.618 \text{ or } .382}$$

② [#14, p.475, §6.4]

With $s = 1/2\, g t^2$, we have

$$v(t) = \text{velocity at time } t$$

$$= ds/dt = gt$$

On the other hand, s and t are related by $2s = gt^2$ or $t = \sqrt{\frac{2s}{g}}$ so as a function of distance

$$v(s) = g\sqrt{\frac{2s}{g}} = \sqrt{2g}\,\sqrt{s}$$

The time interval $[0,T]$ corresponds to the distance interval $[0, S_T]$ with $S_T = \frac{1}{2} g T^2$

The velocity v_T at time T is $gT = \sqrt{2g}\sqrt{S_T}$

Time average of $v = \frac{1}{T}\int_0^T (gt)\,dt = \frac{gT^2}{2T} = \frac{1}{2} gT = \boxed{\frac{1}{2} v_T}$

Distance average of $v = \frac{1}{S_T}\int_0^{S_T}\sqrt{2g}\sqrt{s}\,ds = \sqrt{2g}\;\frac{2}{3}S_T^{3/2}/S_T$

$$= \frac{2}{3}\sqrt{2g}\sqrt{S_T} = \boxed{2/3\, v_T}$$

(3) [#12, p. 491, §6.6]

Let $N(t)$ = # of mosquitoes t weeks into the summer.

We're told that $dN/dt = 2200 + 10e^{0.8t}$

By the Fundamental Theorem of Calculus,

mosquitoes increase between week 5 and week 9

$$= N(9) - N(5) = \int_5^9 \frac{dN}{dt}\, dt$$

$$= \int_5^9 \left(2200 + 10e^{0.8t}\right) dt$$

$$= \left(2200t + \frac{10e^{0.8t}}{0.8}\right)\Bigg|_5^9$$

$$= \boxed{24,860}$$

(4) By Poiseuille's Law

$$F = \text{flux of blood} = \frac{\pi}{8\eta l} P R^4$$

If R_0 and P_0 are the normal values of the radius of arteries and blood pressure and if the constricted values are $R = 3/4 R_0$ and P with F remaining constant, then

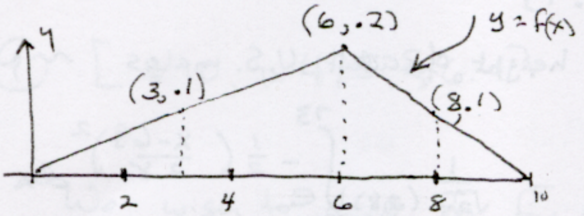
$$\frac{\pi}{8\eta l} P_0 R_0^4 = \frac{\pi}{8\eta l} P\left(\frac{3}{4}R_0\right)^4$$

so $$P = P_0 \frac{R_0^4}{(3/4 R_0)^4} = \left(\frac{4}{3}\right)^4 P_0$$

$$\boxed{P = 3.16 P_0 > 3P_0}$$

i.e., P more than triples from its normal value.

⑤ [# 4, p. 498, § 6.7]



a) $f(x)$ is everywhere ≥ 0 and continuous with

$$\int_{-\infty}^{\infty} f(x)\,dx = \text{area under the graph}$$
$$= \tfrac{1}{2}(6)(.2) + \tfrac{1}{2}(4)(.2)$$
$$= (5)(.2) = 1$$

$\therefore$ $f(x)$ is a probability density function (pdf)

(b) (i) $P(X < 3) = \int_0^3 f(x)\,dx = \frac{1}{2}(3)(.1) = \boxed{.15}$

(ii) $P(3 \leq X \leq 8) = \int_3^8 f(x)\,dx = \frac{1}{2}(.1+.2)3 + \frac{1}{2}(.1+.2)2$

$$= \frac{(.3)5}{2} = \boxed{.75}$$

[the calculation is by the formula for trapezoid areas]

(c) μ = mean of X = $\int_{-\infty}^{\infty} x\, f(x)\,dx = \int_0^6 x\left(\frac{.2}{6}x\right)dx + \int_6^{10} x\left(.2 - \frac{.2}{4}(x-6)\right)dx$

$$= \frac{1}{30}\int_0^6 x^2\,dx + \int_6^{10} x\left(\frac{1}{2} - \frac{x}{20}\right)dx$$
$$= \frac{1}{30}\,\frac{6^3}{3} + \frac{1}{4}\left(10^2 - 6^2\right) - \frac{1}{60}\left(10^3 - 6^3\right)$$
$$= \boxed{5.333}$$

⑥ [# 6, p. 498, § 6.7] By assumption X = lightbulbs lifetime in hours $\sim E(1000)$

(a) (i) $P(\text{bulb fails within 1st 200 hours}) = P(X < 200) = 1 - e^{-200/1000}$
$$= 1 - e^{-1/5} = \boxed{.181}$$

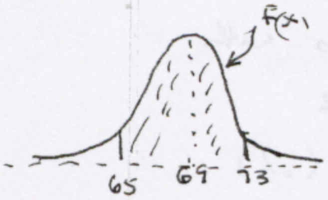
(ii) $P(\text{bulbs last more than 800 hours}) = P(X > 800) = e^{-800/1000}$
$$= e^{-4/5} = \boxed{.449}$$

(b) For m the median lifetime, $\frac{1}{2} = P(X \geq m) = e^{-m/1000} \Rightarrow m = 1000 \ln 2 = \boxed{693}$

⑦ [# 8, p.498, § 6.7]

We're given that X = height of adult U.S. males $\sim N(69, 2.8)$

(a) $P(65 \le X \le 73) = \frac{1}{\sqrt{2\pi}(2.8)} \int_{65}^{73} e^{-\frac{1}{2}\left(\frac{x-69}{2.8}\right)^2} dx$



(on the TI-83) = normalcdf(65, 73, 69, 2.8)

$= \boxed{.847}$

(b) $P(X \ge 72) = \frac{1}{\sqrt{2\pi}(2.8)} \int_{72}^{\infty} e^{-\frac{1}{2}\left(\frac{x-69}{2.8}\right)^2} dx$

on the TI-83 = normal cdf (~~65~~ 72, 80, 69, 2.8)

or any number > 69 + 3(2.8)

$= \boxed{.142}$

So a little over 14% of U.S males are more than 6' tall.

⑧ [#10, p.498, § 6.7]

We're given that X = cereal weight per box in grams $\sim N(\mu, 12)$

(a) If $\mu = 500$, $P(X < 480) = P(X \le 480)$

= normal cdf(450, 480, 500, 12)

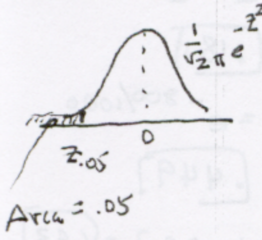
or any number < 500 − 3(12)

$= \boxed{.048}$

(b) We want to set μ at a level for which

$.05 = P(X \le 500) = P\left(\frac{X-\mu}{12} \le \frac{500-\mu}{12}\right) = P\left(Z \le \frac{500-\mu}{12}\right)$

with $Z \sim N(0,1)$



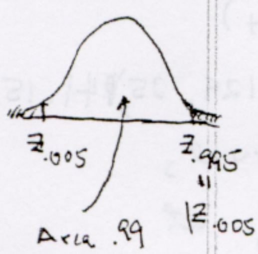
$\therefore \frac{500-\mu}{12} = z_{.05} = \text{invNorm}(.05) = -1.645$

$\mu = 500 + (12)(1.645) = \boxed{519.7} \approx \boxed{520 \text{ grams}}$

SOLUTIONS – HW #6
MATH 1323

(1) [#1, p.13, SPS] We enter the 16 data values for C_{max} in a $\overbrace{\text{list}}^{\text{TI-83}}$, say $L_1 = \{122, \ldots, 60\}$. Then STAT TESTS 8 calls up the TI-83's T Interval program. We highlight DATA on the Input line, enter 2^{nd} L_1 on the List line, 1 on the Frequency line, .95 on the C-Level line, then highlight Calculate and punch Enter. The first line of the output gives the 95% confidence interval $\boxed{(54.6, 105.2)}$ for the mean value μ of the random variable C_{max}

(2) [#2, p.13, SPS] As explained in SPS$^{(*)}$, with $\hat{p} = {}^{1273}/_{1600} \doteq .796$ and $z_{.005} = \text{InvNorm}(.005) = -2.576$, $\hat{p} \pm |z_{.005}| \sqrt{\hat{p}(1-\hat{p})/n}$ are the upper and lower limits of the 99% confidence interval for the true proportion p of U.S. teenage males interested in dating Britney Spears. We can either calculate these limits directly or let the TI-83 calculate them by using the Z Interval program (STAT TESTS 7) with Input highlighting on STATS, entering $\sqrt{\hat{p}(1-\hat{p})}$ for σ, $\hat{p}$ for $\bar{x}$, 1600 for n, and .99 for C-Level. Either way, we get the 99% confidence interval $\boxed{(.770, .822)}$



(*) (i) We can use a Z Interval rather than a T-interval since $n = 1600$ is much larger than 100

(ii) For the same reason, we can assume $\sqrt{\hat{p}(1-\hat{p})} = \sigma$

③ [# 1, p. 25, SPS]

We wish to test the null hypothesis

$H_0: \mu = \mu_0$

against $H_1: \mu \neq \mu_0$

where μ_0 = mean systolic blood pressure among 25-34 year old white males = 124.8

μ = mean systolic blood pressure among 25-34 year old black males

Our sample estimate for μ is 135.2 with the sample standard deviation $s = 13.5$ and sample size $n = 38$. Calling up T Interval and entering the data $\bar{x} = 135.2$, $S_x = 13.5$, $n = 38$, the .95 confidence level is found to be (130.76, 139.64) while the 99% confidence interval is (129.25, 141.15). Since 124.8 is not in either interval, we can $\boxed{\text{reject } H_0 \text{ in favor of } H_1}$ at the 99% confidence level. Actually the 99.9% confidence interval for μ is (127.4, 143.0) so we can still reject H_0 at this level.

(4) [#2, p. 538, § 7.4]

$P(t)$ = # of bacteria cells after t minutes

$= P_0 e^{kt}$ with $P_0 = 60$

(a) We're told that $P(t)$ doubles every 20 minutes

so $120 = P(20) = 60\, e^{20k}$

$2 = e^{20k}$

$$\boxed{k = \frac{\ln 2}{20} = .0347}$$

(b) For any t, $\boxed{P(t) = 60\, e^{.0347\, t}}$

(c) $P(480)$ = # of cells after 8 hours (= 480 min)

$= 60\, e^{(.0347)\,480}$

$= 1{,}00\,663\,2{,}960$

$\approx$ $\boxed{\text{1 billion to 3 figures}}$

(d) $\frac{dP}{dt} = k\, P(t)$ so

$dP/dt\,(480) = (.0347)(100\,663\,2960)$

$\sim$ $\boxed{34{,}9\,00{,}000 \text{ cells/min}}$

e/ $20000 = P(t) = 60\, e^{kt}$

$\Rightarrow$ $kt = \ln(e^{kt}) = \ln \frac{20000}{60} = 5.809$

$\Rightarrow$ $t =$ $\boxed{167.6 \text{ minutes}}$ or $\boxed{2.80 \text{ hours}}$

(5) [# 4, p. 538, § 7.4]

$P(t)$ = # of bacteria after t hours
$= P_0 e^{kt}$

(a) Since $P(2) = 600$ and $P(8) = 75000$

$75000/600 = \cancel{P_0} e^{8k} / \cancel{P_0} e^{2k} = e^{6k}$

$6k = \ln(75000/60)$

$k = \frac{1}{6} \ln(75000/60) = \boxed{.8047}$

Then
$P_0 = \frac{P(2)}{e^{2k}} = \frac{600}{e^{2(.8047)}} = \boxed{120}$ cells

(b) From (a) $\boxed{P(t) = 120\, e^{.8047 t}}$

(c) $P(5) = 120\, e^{5(.8047)} = \boxed{6707}$ cells

(d) $\left(\frac{dP}{dt}\right)(5) = k\, P(5) = (.8047)(6707) = \boxed{5398 \text{ cells/hour}}$

(e) $200000 = P(t) = 120\, e^{kt}$

$.8047\, t = kt = \ln \frac{200000}{120} = 7.4186$

$\boxed{t = 9.219 \text{ hours}}$

(6) [#6, p. 538, §7.4]

We can build ~~[illegible]~~ an exponential model $P(t) = P_0 e^{kt}$ for the US population in year t from any two values for $P(t)$

(a) Using $P(1900) = 76$ million, $P(1910) = 92$ million

we derive a model with

$$e^{10k} = \frac{P_0 e^{1910k}}{P_0 e^{1900k}} = 92/76$$

With this ~~[illegible]~~ model

$$P(2000) = P(1900 + 100) = P(1900)\, e^{100k}$$

$$= (76)\,(92/76)^{10}$$

$$= 513.5 \text{ million}$$

This is way off from the actual value $P(2000) = 275$ million

Explanation: Fast growth in 1900-1910 due to largescale immigration

Slow growth in war ~~[illegible]~~ decades

1900-1910 wasn't a "typical" decade

(b) Using $P(1980) = 227$ and $P(1990) = 250$

we obtain a model with

$$e^{10k} = \frac{250}{227}$$

This model predicts

$$P(2000) = P(1990)\,e^{10k} = \boxed{275.3 \text{ million}}$$

[comparing well with the actual value]

$$P(2010) = P(1990)\,(e^{10k})^2 = \boxed{303.2 \text{ million}}$$

$$P(2020) = P(1990)\,e^{30k} = (250)\left(\frac{250}{227}\right)^3$$

$$= \boxed{334.0 \text{ million}}$$

(c) Model 1: $e^{10k} = 92/76 = 1.210$

$P(t+1900) = 76\, e^{kt} = 76\,(1.21)^{t/10}$

Model 2. $e^{10k} = \frac{250}{227} = 1.101$

$P(t+1980) = 227\,(1.101)^{t/10}$

$P(1900) = 227\,(1.101)^{-80/10} = 164.9$

$P(t+1900) = 164.9\,(1.101)^{t/10}$

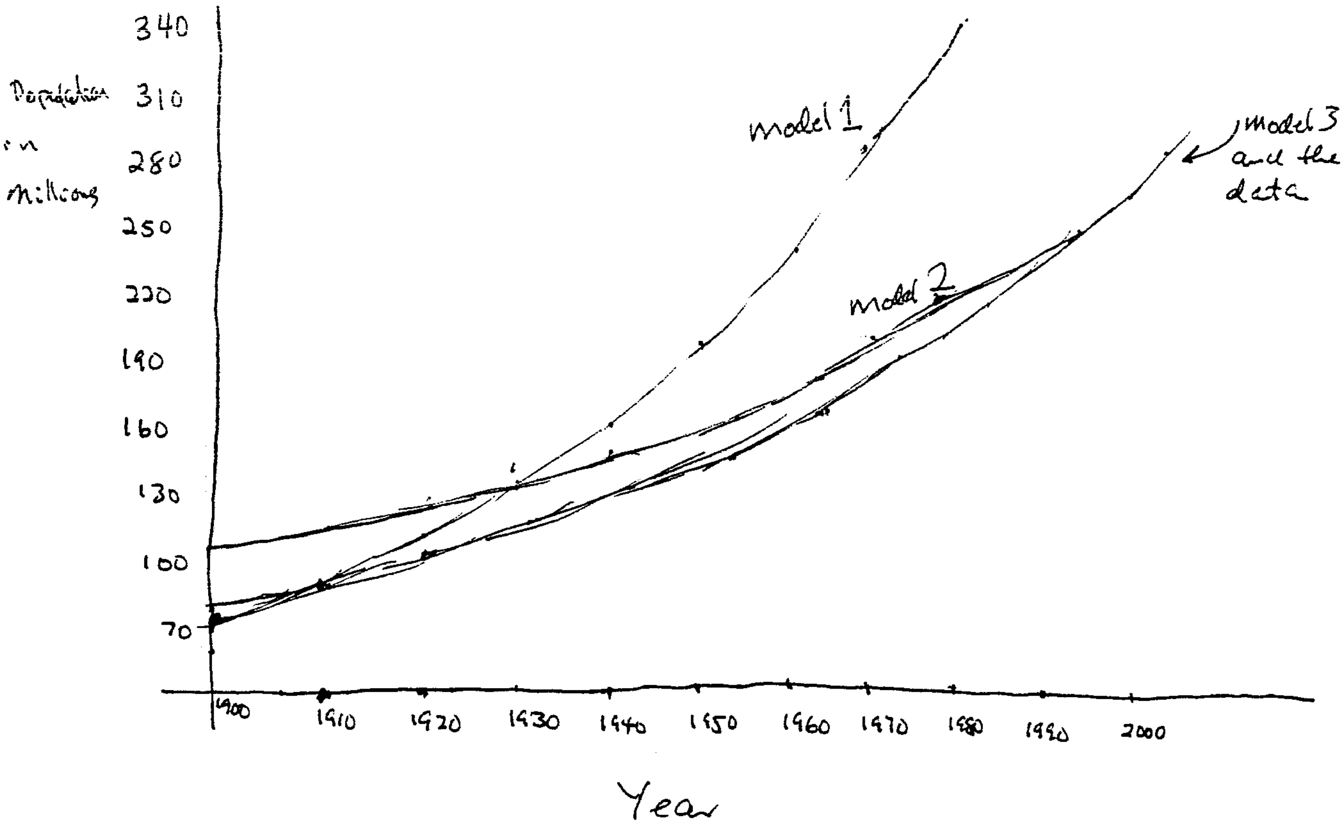
Model 3: Using LnReg to get the best straight line fit of the data pairs $(t, \ln P(t))$ leads to the model

$P(t+1900) = 80.45\,(1.136)^{t/10}$ with

$r^2 = .9938$. This is a great fit.

Year	Model 1	Model 2	Model 3	Actual
1900	76	164.9	80.45	76
1910	92	115.5	91.4	92
1920	111.4	127.2	103.8	106
1930	134.8	140.0	117.9	123
1940	163.2	154.1	134.0	131
1950	197.5	169.7	152.2	150
1960	239.1	186.9	172.9	179
1970	289.5	205.7	196.4	203
1980	350.4	226.5	223.1	227
1990	424.2	249.4	253.5	250
2000	513.5	275.6	287.9	275

Conclusion: Both Models 1 and 2 don't correspond well with the data. Model 3 corresponds very well


340
Population
310
in
280
Millions
250
220
190
160
130
100
70
model 1
model 2
model 3
and the
data
1900
1910
1920
1930
1940
1950
1960
1970
1980
1990
2000
Year