Solutions — HW # 7
Math 1323

① [#18, p. 528, §7.3]

$$e^{-y}\, dy/dx + \cos x = 0$$

$$\int -e^{-y}\, dy = \int \cos x \, dx$$

$$e^{-y} = \sin x + C$$

$$\boxed{y = -\ln(\sin x + C)} = \ln\left(\frac{1}{\sin x + C}\right)$$

Comments: (i) y isn't defined for $\sin x + C \le 0$ so we must have $C > -1$ in order for y to be defined on intervals and $C > 1$ in order for y to be everywhere defined

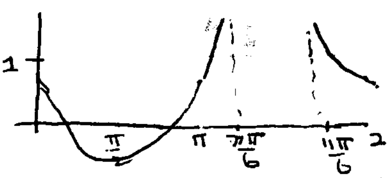
(ii) It's enough to graph $y(x)$ for $0 \le x \le 2\pi$ since the value of $y(x)$ doesn't change when we add on an integer multiple of 2π to x

(iii) At $x = 0, \pi$, or 2π, $y = -\ln C$. When $C > 1$ the maximum value of y occurs at $\frac{3\pi}{2}$ and the minimum value occurs at $\pi/2$

$C = 0$ $(y = -\ln \sin x)$

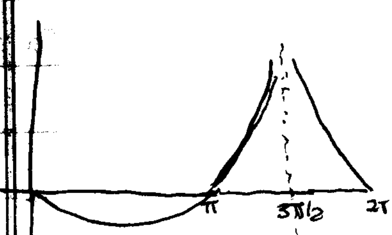
[y isn't defined for $\pi \le x \le 2\pi$]

$C = 1/2$ $(y = -\ln(\sin x + 1/2))$



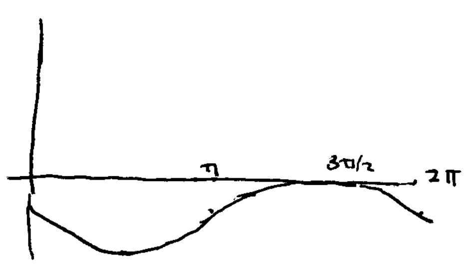
[y isn't defined between $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$]

$C = 1$ $(y = -\ln(\sin x + 1))$



[y isn't defined at $\frac{3\pi}{2}$]

$C = 2$ $(y = -\ln(\sin x + 2))$



② [#28, p. 528, § 7.3]

$T(t)$ = temperature in degrees Celsius at time t

$T(0) = 95^\circ$

Room temperature $= 20^\circ$

By Newton's law of cooling $\frac{dT}{dt} = -k(T-20)$

or $\frac{d}{dt}(T-20) = -k(T-20)$

so $T-20 = (T(0)-20)e^{-kt} = 75e^{-kt}$

$$\boxed{T = 20 + 75e^{-kt}}$$

If we assume $dT/dt = -1$ when $T=70^\circ$ as in #28 § 7.2,

then $-1 = -k(70-20) = -50k$ so $k = \frac{1}{50}$

and $T(t) = 20 + 75e^{-t/50}$

This gives $T(10) = 20 + 75e^{-2} = 30.2^\circ$

and $T(20) = 20 + 75e^{-4} = 21.4^\circ$ ≈ room temperature

which sounds about right [lukewarm coffee after 10 minutes, cold coffee after 20 minutes]

③ [#30, p. 528, § 7.3]

Assumptions : $x(t)$ = # of moles of C at time t

$x(0) = 0$

$\frac{dx}{dt} = k(a-x)(b-x)$ with $k>0$

with a = initial # of moles of A

b = " " " " " B

(a) Assume $a \neq b$. Then $\frac{(a-b)\,dx}{(x-a)(x-b)} = (a-b)k\,dt$ (*)

Since. $\frac{a-b}{(x-a)(x-b)} = \frac{1}{x-a} - \frac{1}{x-b}$

integrating (*) on both sides gives

$$\ln\frac{(x-a)}{x-b} = (a-b)kt + K$$

Substituting $t=0$ and $x=0$, we obtain $K = \ln a/b$

Exponentiating $\quad \dfrac{x-a}{x-b} = \dfrac{a}{b} e^{(a-b)kt}$

~~which~~ or $\quad x\left(1 - a/b\, e^{(a-b)kt}\right) = a - a e^{(a-b)kt}$

so $\quad x(t) = \boxed{a\left\{\dfrac{1 - e^{(a-b)kt}}{1 - \frac{a}{b} e^{(a-b)kt}}\right\}}$

[can be rewritten many ways including

$$x(t) = b\left\{\frac{1 - e^{(b-a)kt}}{1 - \frac{b}{a} e^{(b-a)kt}}\right\} \;]$$

Note: With $b > a$, $e^{(a-b)kt} = e^{-(b-a)kt} \to 0$ as $t \to \infty$

so $\quad x(t) \to a \quad$ as $\quad t \to \infty$.

This makes sense. With $b > a$, the reaction uses up all of compound A and then slows to a halt with a final value of a moles of C and with $b-a$ moles of B left over

Similarly for $a > b$, $\quad x(t) \to b$ as $t \to \infty$, etc

(b) With $a = b$, we have $\quad \dfrac{dx}{(x-a)^2} = k\,dt$

$$-\left(\frac{1}{x-a}\right) = kt + K$$

and substituting $x=0, t=0$, $K = \frac{1}{a}$

Then $\quad \dfrac{1}{x-a} = -\left(kt + \frac{1}{a}\right) = -\left(\dfrac{akt+1}{a}\right)$

or

$$\boxed{x = a - \left(\frac{a}{akt+1}\right)}$$

If $x = a/2$ when $t = 20$, then $\dfrac{a}{2} = \dfrac{a}{20ka+1}$

or $20ka + 1 = 2$, $ak = \frac{1}{20}$ and

$$x = a\left\{1 - \left(\frac{1}{\frac{t}{20}+1}\right)\right\} = \frac{at/20}{t/20+1} = \boxed{\frac{at}{t+20}}$$

④ [# 34, p. 529, §7.3]

Let $x(t)$ = # of billions of new currency in circulation ~~at~~ t days after the currency change decision

(a) Then $\frac{10-x}{10}$ is the fraction of ~~new~~ old currency in circulation. Since \$50 million = \$.05 billion, $(.05)\left(\frac{10-x}{10}\right)$ is the amount of old currency replaced by new currency on day t

This gives us $\frac{dx}{dt} = (.05)\left(\frac{10-x}{10}\right) = .005\,(10-x)$
$= -(.005)(x-10)$

(b) With $y(t) = x(t) - 10$, $y(0) = -10$ and $\frac{dy}{dt} = -.005\,y$

so $y(t) = -10\,e^{-.005t}$ and

$$\boxed{x(t) = 10 - 10e^{-.005t}}$$

(c) $x(t) = 9$ implies $-1 = -10\,e^{-.005t}$

or $\frac{1}{10} = e^{-.005t}$

$\Rightarrow t = \frac{\ln 10}{.005} = \boxed{460.5 \text{ days}}$

⑤ [# 4, p. 548, §7.5] (a) Let $L_1 = \{0, 2, \ldots, 18\}$ and $L_2 = \{18, 39, \ldots, 672\}$ be the lists of times (in hours) and yeast cell data. For $x(t)$ = # of yeast cells after t hours, ExpReg L_1, L_2 gives the exponential prediction model $x(t) = 35.6\,(1.22)^t$

An alternative way to obtain this prediction equation is to create $L_3 = \ln(L_2)$ and then use LinReg $(a+bx)$ L_1, L_2 to obtain the prediction equation $\ln x(t) = 3.572 + .203\,t$ or

$$x(t) = e^{3.572}\left(e^{.203}\right)^t = 35.6\,(1.22)^t$$

The TI-83 reports the value of R^2 as .858

[not bad but also not great]

(b) For a logistic model, we use the prediction equation

$$x(t) = \frac{K}{1+\left(\frac{K-18}{18}\right)e^{-kt}}$$

We can estimate k from $x(t) \approx x(0)e^{kt} = 18e^{kt}$ for small t and $x(2) = 39$. Thus $\frac{39}{18} \approx e^{2k}$ or $k \approx \frac{1}{2}\ln\frac{39}{18} = .387$. From the data, $x(t)$ is "leveling off" for $t \geq 16$ with $K = \lim_{t\to\infty} x(t)$ likely to be in the range from 675 to 685. If we take $K = 680$ our logistic prediction equation is $x(t) = \dfrac{680}{1+36.8e^{-.387t}}$

(c)

t	Exp. model $x(t) = 35.6(1.22)^t$	Data Value for $x(t)$	Logistic Prediction with $K = 680$	Error in logistic prediction
0	35.6	18	18.0	0.00
2	52.9	39	37.8	1.2
4	78.9	80	77.0	2.0
6	117.38	171	147.5	23.5
8	174.7	336	255.2	80.8
10	260.0	509	384.7	124.3
12	387.0	597	502.2	94.8
14	576.1	640	584.6	55.4
16	857.4	664	632.4	31.6
18	1276.2	672	657.2	14.8

Using 1-Var Stats, $SSX = \Sigma(\text{datavalue} - \text{datamean})^2 = 679640$ while SSE = sum of squares of logistic prediction errors = 35810 giving $R^2 = 1 - SSE/SSX = .35810/679640 = .947$

An R^2 this high is considered to be a very good fit. Even so, the middle range errors are sizeable. On the other hand, the exponential model errors are clearly much larger.

[# 6, pp 548-9, §7.5]

(6) Using $k = \frac{1}{10} \ln 1.136$ the logistic model for US population is

$P(t)$ = Population in millions t years after 1990

$P(0) = 250$

$dP/dt = kP(1 - P/K)$

with solution

$$P(t) = \frac{K}{1 + \left(\frac{K-250}{250}\right) e^{-kt}} = \frac{K}{1 + \left(\frac{K-250}{250}\right)(1.136)^{-t/10}}$$

Predicted populations

Year	K = 1000 (1 billion)	K = 5000 (5 billion)
2100 (t=110)	575 million	881 million
2200 (t=210)	829 million	2.17 billion

(7) [# 8, p. 549, §7.5]

(a) For t small, we expect $P(t)$ = # of fish after t years to grow exponentially

$$P(t) \approx P_0 e^{kt} = 400 e^{kt}$$

Since $P(1) = 1200$, $e^k = \frac{1200}{400}$

or $k = \ln 3$

With a carrying capacity K of 10000 fish the logistic equation $dP/dt = kP(1 - \frac{P}{K})$ has the solution

$$P(t) = \frac{10000}{1 + \left(\frac{10000 - 400}{400}\right) e^{-\ln 3 t}} = \boxed{\frac{10000}{1 + 24 e^{-1.10t}}}$$

(b) $P(t) = 5000 \Rightarrow 1 + 24 e^{-1.10t} = \frac{10000}{5000} = 2$

$-1.10t = \ln(1/24)$

$$\boxed{t = 2.89 \text{ years}}$$

MATH 1323
HW #8
SOLUTIONS

(1) [# 22, p. 521, § 7.2]

We wish to find $y(1)$ where $y'(x) = 1 - x\,y(x)$, $y(0) = 0$

(a) Euler's Method with Step Size $h = 0.2$:

$$\begin{aligned}
x_0 &= 0 & y_0 &= 0 \\
x_1 &= .2 & y_1 &= y_0 + .2(1 - x_0 y_0) = .2 \\
x_2 &= .4 & y_2 &= y_1 + .2(1 - x_1 y_1) = .392 \\
x_3 &= .6 & y_3 &= y_2 + .2(1 - x_2 y_2) = .561 \\
x_4 &= .8 & y_4 &= y_3 + .2(1 - x_3 y_3) = .693 \\
x_5 &= 1.0 & y(1) \approx y_5 &= y_4 + .2(1 - x_4 y_4) = \boxed{.782}
\end{aligned}$$

(b) $y' + xy = 1$

multiplying both sides by $e^{\int x\,dx} = e^{x^2/2}$

$$\frac{d}{dx}\left(y e^{x^2/2}\right) = e^{x^2/2}$$

$$y(1)e^{1/2} - \cancel{y(0)}e^0 = y e^{x^2/2}\Big|_0^1 = \int_0^1 \frac{d}{dx}\left(y e^{x^2/2}\right) = \int_0^1 e^{x^2/2}\,dx$$

$y(0) = 0$

$$y(1) = e^{-1/2} \int_0^1 e^{x^2/2}\,dx$$

(using FnInt) $= \boxed{.725}$

(2) [# 24, p. 521, § 7.2] We wish to find $y(1.4)$ where

$$y'(x) = x - x\,y(x) = x(1 - y(x)) \quad \text{with } y(1) = 0$$

(a) Euler's Method with Step Size $h = 0.2$

$$\begin{aligned}
x_0 &= 1 & y_0 &= 0 \\
x_1 &= 1.2 & y_1 &= y_0 + .2(x_0 - x_0 y_0) = .2 \\
x_2 &= 1.4 & y(1.4) \approx y_2 &= y_1 + .2(x_1 - x_1 y_1) = \boxed{.392}
\end{aligned}$$

(b) Using the method of separable equations

$$\frac{dy}{1-y} = x\,dx$$

$$-\ln(1-y) = x^2/2 + C \quad \text{with } 0 = 1/2 + C \text{ or } C = -\tfrac{1}{2}$$

$$-\ln(1 - y(1.4)) = \frac{(1.4)^2 - 1}{2} = .48$$

$$1 - y(1.4) = e^{-.48} \quad \text{or} \quad y(1.4) = 1 - e^{-.48} = \boxed{.3812}$$

EXTRA CREDIT:
To compute $y(1.4)$ to 4 decimal places by Euler's method requires a step size around .0005

(3) (i) $\frac{dy}{dt} + 3y = t$

Step 1: $\frac{d}{dt}(ye^{3t}) = y'e^{3t} + y(3e^{3t}) = te^{3t}$

Step 2: $ye^{3t} = C + \int te^{3t}dt \underset{\text{(parts)}}{=} C + \frac{te^{3t}}{3} - \frac{e^{3t}}{9}$

Step 3: $\boxed{y = Ce^{-3t} + t/3 - 1/9}$

Check: $\frac{d}{dt}\{Ce^{-3t} + (t/3 - 1/9)\} + 3\{Ce^{-3t} + (t/3 - \frac{1}{9})\}$

$= \cancel{-3Ce^{-3t}} + \cancel{1/3} + \cancel{3Ce^{-3t}} + t \cancel{- 1/3} = t$ ✓

(ii) $\frac{dy}{dt} + 2y = \sin t$

Step 1: $\frac{d}{dt}(ye^{2t}) = y'e^{2t} + y(2e^{2t}) = e^{2t}\sin t$

Step 2: $ye^{2t} = C + \int e^{2t}\sin t\, dt$

(by parts twice or a table) $= C + \frac{2}{5}e^{2t}\sin t - \frac{1}{5}e^{2t}\cos t$

Step 3: $\boxed{y = Ce^{-2t} + \frac{2}{5}\sin t - \frac{1}{5}\cos t}$

Check: $\frac{d}{dt}\{Ce^{-2t} + \frac{2}{5}\sin t - \frac{1}{5}\cos t\} + 2\{Ce^{-2t} + \frac{2}{5}\sin t - \frac{1}{5}\cos t\}$

$= \cancel{-2Ce^{-2t}} + \cancel{\frac{2}{5}\cos t} + \frac{1}{5}\sin t + \cancel{2Ce^{-2t}} + \frac{4}{5}\sin t - \cancel{\frac{2}{5}\cos t}$

$= \sin t$ ✓

Alternative Method: The general solution is $Ce^{-2t} + Q(t)$ for $Q(t)$ any particular solution. We guess that $Q(t)$ will have the form $A\sin t + B\cos t$

Then $\frac{dQ}{dt} + 2Q = A\cos t - B\sin t + 2A\sin t + 2B\cos t$

$= (A + 2B)\cos t + (2A - B)\sin t$

Putting $2A - B = 1$, $A + 2B = 0$

we solve to obtain $A = 2/5$, $B = -1/5$

(iii) $\frac{dy}{dt} + y = e^{2t}$

Step 1: $\frac{d}{dt}(ye^{t}) = y'e^{t} + ye^{t} = e^{3t}$

Step 2: $ye^{t} = C + \frac{1}{3}e^{3t}$

Step 3: $\boxed{y = Ce^{-t} + \frac{1}{3}e^{2t}}$

Check: $\frac{dy}{dt} + y = \cancel{-Ce^{-t}} + \frac{2}{3}e^{2t} + \cancel{Ce^{-t}} + \frac{1}{3}e^{2t} = e^{2t}$ ✓

(iv) $\frac{dy}{dt} + y = e^t \sin t$

Step 1: $\frac{d}{dt}(ye^t) = y'e^t + ye^t = e^{2t}\sin t$

Step 2: $ye^t = C + \int e^{2t}\sin t \underset{[\text{as in (ii)}]}{=} C + \frac{2}{5}e^{2t}\sin t - \frac{1}{5}e^{2t}\cos$

Step 3: $\boxed{y = Ce^{-t} + 2/5\, e^t \sin t - \frac{1}{5}e^t \cos t}$

Alternative: Guess a particular solution of the form $Q(t) = Ae^t\sin t + Be^t\cos$ and, as in (ii), solve equations to find A and B

Ck: $\frac{d}{dt}\left\{Ce^{-t} + \frac{2}{5}e^t\sin t - \frac{1}{5}e^t\cos t\right\} + \left\{Ce^{-t} + \frac{2}{5}e^t\sin t - \frac{1}{5}e^t\cos t\right\}$

$= \cancel{-Ce^{-t}} + (2/5 + 1/5)e^t\sin t + \cancel{(2/5 - \frac{1}{5})e^t\cos t}$

$\cancel{+Ce^{-t}} + 2/5\, e^t\sin t \cancel{- 1/5\, e^t\cos t}$

$= e^t\sin t$ ✓

(v) $\frac{dy}{dt} + y/t = 4t + 1$

Step 1: Noting that $e^{\int dt/t} = e^{\ln t} = t$

$\frac{d}{dt}(yt) = y't + y = (y' + y/t)t = 4t^2 + t$

Step 2: $yt = C + \int(4t^2 + t)\,dt = C + 4/3\, t^3 + t^2/2$

Step 3: $y = C/t + 4/3\, t^2 + t/2$

Check: $\frac{dy}{dt} + y/t = \cancel{-C/t^2} + 8/3\, t + 1/2 + \cancel{C/t^2} + \frac{4}{3}t + \frac{1}{2}$

$= \frac{12}{3}t + 1 = 4t + 1$ ✓

(vi) $\frac{dy}{dt} + 3y/t = t^2 + 1$

Step 1: Noting that $e^{\int 3/t\, dt} = e^{3\ln t} = t^3$

$\frac{d}{dt}(yt^3) = y't^3 + y(3t^2) = (y' + \frac{3}{t}y)t^3 = t^5 + t^3$

Step 2: $yt^3 = C + \int(t^5 + t^3)\,dt = C + \frac{1}{6}t^6 + \frac{1}{4}t^4$

Step 3: $y = C/t^3 + 1/6\, t^3 + 1/4\, t$

Check: $\frac{dy}{dt} + 3y/t = \cancel{-3C/t^4} + \frac{1}{2}t^2 + \frac{1}{4} + \cancel{\frac{3C}{t^4}} + \frac{1}{2}t^2 + \frac{3}{4}$

$= t^2 + 1$ ✓