MATH 4111 HOMEWORK ASSIGNMENT #4
Due Thursday, October 6

I. Problems on Norms for Linear Transformations. Let $\mathbb{F}$ =$\mathbb{R}$ or $\mathbb{C}$. Suppose (V,$\|\cdot\|$) and (V$'$,$\|\cdot\|'$) are normed vector spaces over $\mathbb{F}$. For each linear transformation T:V→ V$'$, let $\|T\|_{op} = \sup\{\|T(u)\|' : u \in V \text{ with } \|u\| = 1\}$. Notice that, if $\|T\|_{op} < \infty$, then, for any $v \in V$, $v = ||v||u$ for some u with $||u||$=1 and $\|T(v)\|' = ||(||v||\, T(u))||' = ||v||\, ||T(u)||' \leq \|T\|_{op}\, \|v\|$. Hence, T is uniformly continuous from (V,$\|\cdot\|$) into (V$'$,$\|\cdot\|'$), since for v and w in V with $\|v - w\| < \delta_\epsilon = \epsilon/\|T\|_{op}$, we have $\|T(v) - T(w)\|' = ||T(v - w)||' < \epsilon$.

1. If T:V→V$'$ is linear, show that $\|T\|_{op} < \infty \Leftrightarrow$ T is continuous at 0 in the sense that, for each ϵ>0, we have some δ_ϵ>0 s.t.$\|T(v)\|' = \|T(v) - T(0)\|' < \epsilon$ when $\|v\| = ||v - 0|| \leq \delta_\epsilon$. (in view of the above remarks, it's only necessary to prove $\Leftarrow$).

2. For obvious reasons, we say a linear transformation T:V→V$'$ is <u>bounded</u> (relative to $\|\cdot\|$ and $\|\cdot\|'$) if $\|T\|_{op}$<∞. Show that $\|\cdot\|_{op}$ is a norm on the vector space $\mathcal{B}$(V,V$'$) consisting of all bounded linear operators from (V,$\|\cdot\|$) into (V$'$, $||\cdot||'$).

If (V$'$, $||\cdot||'$) = (V,$\|\cdot\|$), we write $\mathcal{B}$(V) for $\mathcal{B}$(V,V). Show that, for S and T in $\mathcal{B}$(V), the composition S∘T is also in $\mathcal{B}$(V) and $\|S\circ T\|_{op} \leq ||S||_{op}||T||_{op}$. It's customary in linear algebra to write ST for S∘T.

3. When (V,$\|\cdot\|$) is complete and T $\in \mathcal{B}$(V), show that $\exp(T) = e^T = \sum_{k=0}^{\infty} \frac{1}{k!}T^k$ defines a member of $\mathcal{B}$(V) whose operator norm is $\leq e^{||T||_{op}}$. This comes down to checking that, for any $v \in V$, $w_n = \sum_{k=0}^{n} \frac{1}{k!}T^k\,(v)$ defines a Cauchy sequence $(w_n)_{n\geq 0}$ in the complete metric space (V,$d_{||\cdot||}$) and, with w the limit of this sequence, that $\|w_n\| \leq e^{||T||_{op}}||v||$ for each n. In checking this, note that $||T^k v|| \leq ||T||_{op}^k\, ||v||$ for each $k \geq 0$ by a trivial induction argument..

4. If T $\in \mathcal{B}$(V,V$'$) happens to be 1-1 and onto, don't bother to repeat the easy argument in linear algebra showing that T^{-1} is linear from V$'$ back to V. Show that T^{-1} is bounded from $(V', ||\cdot||')$ into $(V, ||\cdot||)$ $\Leftrightarrow$ $\inf\{\|T(u)\| : u \in V \text{ with } \|u\| = 1\} > 0$.

Remarks : When $f:\mathbb{R}\to\mathbb{R}$ is continuous, the Fundamental Theorem of Calculus tells us that $\mathcal{D}\mathcal{I}f = f$ where $\mathcal{D}=d/dx$ is the differentiation operator and $(\mathcal{I}f)(x) = \int_0^x f(t)dt$ is the anti-derivative of f whose value at 0 is 0. Obviously, $\mathcal{D}$ and $\mathcal{I}$ are linear operators. But $\mathcal{I}$ is much better behaved than $\mathcal{D}$. Thus, for any finite $a, \mathcal{I}$ maps $C(-a,a) = \{f \in C(\mathbb{R})$: f is zero off the interval $(-a,a)\}$ continuously into BC($\mathbb{R}$) ={bounded members of C($\mathbb{R}$} since standard easy properties of Riemann integrals yield $||\mathcal{I}f||_\infty \leq 2a||f||_\infty$. On the other hand, there are no choices of norms relative to which $\mathcal{D}$ is continuous; $e.g.$, there's no fixed constant c for which $\|\mathcal{D}f||_\infty \leq c||f||_\infty$ for every differentiable function f. This is just one of many ways in which integral calculus has far superior properties to those of differential calculus. In Math 4151, the theories of differential and integral calculus are extended to functions defined on a smooth manifold (every such manifold can be realized as a smooth surface in $\mathbb{R}^N$ for some (usually large) N $\in \mathbb{N}$. The integral calculus theory is easy and best explained using norms. The differential calculus theory is far from easy, highly technical, and norms are not useful. Sadly, general relativity theories rely mostly on manifold derivatives and are therefore not easy to comprehend.

Many of you may have seen exponentials of matrices and have heard a little bit about their usefulness for solving systems of linear differential equations. Problem I-3 generalizes matrix exponentials. This generalization is useful in several applied areas.

II. Problems on Matrix Norms. Again let $\mathbb{F}$=$\mathbb{R}$ or $\mathbb{C}$. For each $m, n \in \mathbb{N}$, denote by $\mathbb{F}^{m\times n}$ the mn-dimensional vector space of all $m \times n$ matrices A with entries $A_{i,j} \in \mathbb{F}$. (Some authors prefer to denote the i,j entry of A by A(i,j) in order to emphasize that A can be regarded as a function from $\{1,2,\cdots,m\} \times \{1,2,\cdots,n\}$ into $\mathbb{F}$). As usual, for each A $\in \mathbb{F}^{m\times n}$,

$v \mapsto \mathrm{L}_A(v) = \mathrm{A}v$ is a linear transformation from $\mathbb{F}^{n\times 1}$ into $\mathbb{F}^{m\times 1}$ and, conversely, every linear transformation T: $\mathbb{F}^{n\times 1} \to \mathbb{F}^{m\times 1}$ is of the form L_A for a unique $\mathrm{A} \in \mathbb{F}^{m\times n}$ called the matrix of T. By showing that $||\mathrm{L}_A||_{op}$ is finite for some choices of norms and using the fact to be proved later that any two norms on a finite dimensional space are equivalent, it follows that any linear transformation between two finite dimensional normed linear spaces is uniformly continuous (no surprise).

1. Calculate $||\mathrm{L}_A||_{op}$ relative to the uniform norms $\|\cdot\|_\infty$ on $\mathbb{F}^{n\times 1}$ and $\mathbb{F}^{m\times 1}$.

2. Using the taxi-cab norms $\|\cdot\|_1$ on $\mathbb{F}^{n\times 1}$ and $\mathbb{F}^{m\times 1}$, show that $||\mathrm{L}_A||_{op} \leq \sum\limits_{i,j}|\mathrm{A}_{i,j}|$.

3. Using the Euclidean norms $\|\cdot\|_2$ on $\mathbb{F}^{n\times 1}$ and $\mathbb{F}^{m\times 1}$, show that $||\mathrm{L}_A||_{op} \leq |||A||| = \left(\sum\limits_{i,j}|\mathrm{A}_{i,j}|^2\right)^{1/2}$. It's customary to call $|||\cdot|||$ the Hilbert-Schmidt norm on $\mathbb{F}^{m\times n}$ although it could be called the Euclidean norm and could be denoted by $\|\cdot\|_2$.

III. Problems on Arc length metrics

1. Recall that, for $t \mapsto p(t)$ piecewise continuously differentiable from [0,1] into $\mathbb{R}^n$, <u>the arc length along</u> the path p is defined to be $L_p = \int_0^1 ||p'(t||_2\, dt$ and it's simple to show that $L_p \geq \|p(1) - p(0)||_2$ with equality $\Leftrightarrow$ $p(t) = p(0) + f(t)(p(1) - p(0))$ where f is piecewise continuously differentiable and monotonic increasing from [0,1] onto [0,1]. When S is a $k-$dimensional smooth surface in $\mathbb{R}^n$, the arc length metric d_{arc} on S is defined by $d_{arc}(a,b) = inf\{L_p : p$ is a piecewise continuously differentiable path with $p(t) \in$ S $\forall t \in [0,1]$ and $p(0) = a,\ p(1) = b\}$. Show that (S,d_{arc}) is a metric space.

2. The unit sphere S^n about 0 in $\mathbb{R}^{n+1}$ is defined to be $\{a \in \mathbb{R}^{n+1} : ||a||_2 = 1\}$. Show that the arc length metric on S^n is equivalent to the restriction to $\mathrm{S}^n \times \mathrm{S}^n$ of the Euclidean metric d_2 on $\mathbb{R}^n$. Explicitly, find the smallest possible numbers $c,\ c'$ for which

$$\tfrac{1}{c'}d_{arc}(a,b) \leq d_2(a,b) = ||b-a||_2 \leq c d_{arc}(a,b)$$

for all points $a, b\, in$ S^n.

Hints: When $b \notin \{a, -a\}$, there is a unique $\theta_1 \in (0, \pi)$ and a unique unit vector u perpendicular to a such that $b = cos(\theta_1)a + sin(\theta_1)u$. For $t \mapsto p(t)$ continuously differentiable from $[0,1]$ into S^n with $p(0) = a,\ p(1) = b$, check that $p(t) = cos\,(\theta(t))\,a + sin(\theta(t)q(t)$ where $t \mapsto \theta(t)$ is differentiable from $[0,1]$ into $[0,\pi)$ with $\theta(0) = 0, \theta(1) = \theta_1$, and $t \mapsto q(t)$ is differentiable from $[0,1]$ into the space of vectors in $\mathbb{R}^{n+1}$ which have unit length and are perpendicular to a with $q(1) = u$. Deduce from this that $L_p \geq \int_0^1 |\theta'(t)|dt \geq \theta_1$ with equality $\Leftrightarrow$ $\theta'(t) \geq 0$ and $q(t) = u$ for all $t \in [0,1]$. This argument proves that $d_{arc}(a,b) = \theta_1$ is the socalled "great circle distance" from a to b. By a trivial limiting argument, $d_{arc}(a, -a) = \pi$. Then finish off Problem 2 using standard trig identities or, if you prefer, standard Euclidean plane geometry results comparing the chord length with the circular arc length for two points on a circle of radius 1.

You could carry out the above inequalities using the n − dimensional version of spherical coordinates but this is very strongly not recommended since the coordinates just get in the way of the easy coordinate − free calculation.

3. Give an example of a 2-dimensional smooth surface $\mathrm{S} \subset \mathbb{R}^3$ where the arc length metric is not equivalent to to the restriction of the Euclidian metric.

Hint: Your surface will need to come very close to "folding back on itself" in order to have pairs of points whose arc length distances are large but whose Euclidean distances are tiny. Don't bother to concoct an explicit parametrization of your surface. Instead, settle for a sketch (as nicely drawn as your artistic talents will permit) accompanied by a short explanation how the surface could be constructed.