[Recall that Part I of the Final Exam was to write an essay on one of three topics with material for the essay to be drawn from class notes or the textbook.]

Part II. Give solutions for five of the following nine problems. Be sure that your method is clearly indicated so that, if you make some algebraic mistakes, you can still receive a large amount of partial credit.

II(1). Let $S$ be the semi-circular region in the first quadrant of $\mathbb{R}^2$ bounded by the line segment joining 0 and 2 and the circle of radius 1 about 1. Describe a composition $F$ of a Möbius transformation, a power function, and another Möbius transformation for which $F$ maps $S$ conformally and one-to-one onto $\mathbb{D}$. Identify explicitly the three factors of $F$ but don't bother to grind out the formula for their composition.

**Solution:** One possibility for $F$ is $T_3 \circ p_2 \circ T_1$ where $T_1(z) = \left[ z, 0, 1, 2 \right] = \frac{z}{2-z}$ is a Möbius transformation mapping $S$ to the first quadrant $S_1$, $p_2(z) = z^2$ is the power function mapping $S_1$ to the upper half plane $\mathbb{H}$, and $T_3(z) = \frac{z-i}{z+i}$ is the Cayley transform mapping $\mathbb{H}$ to $\mathbb{D}$.
II(2) Suppose $f$ is holomorphic on a domain $D$ containing $\overline{D}$ and $f(\partial D) = \partial D$. Use the maximum modulus theorem to show that $f(D) \supset D$.

**Solution.** By the maximum modulus principle, the maximum value of $|f|$ on $\overline{D}$ occurs on $\partial D$ and is therefore equal to 1, giving $|f| < 1$ on $D$ since $f(\partial D) = \partial D$ implies $f$ isn't a constant function. One can make a somewhat awkward argument that $f(D) = D$ using the open mapping theorem. Easier is to note that $|f| < 1$ dictates that $0 \in f(D)$ for otherwise $1/f$ would satisfy the same hypotheses as $f$ and have magnitude $> 1$ on $D$. For $z_0 \in D$ and $T_{z_0} = \frac{z - z_0}{z_0 z - 1}$ the Möbius transformation mapping $D$ to itself and interchanging $0$ and $z_0$, the above argument gives $0 \in (T_{z_0} \circ f)(D)$ which implies $z_0 \in f(D)$. Hence $f(D) = D$.

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II(3) Let $f(z) = \frac{1}{\cosh(\pi z)}$.

(i) Show that $\{z_k = i(k + 1/2) : k \in \mathbb{Z}\}$ is the set of poles of $f$ and that all of these poles are simple, i.e., have order 1. Compute $\text{Res}(f, z_k)$; you should discover that the result depends only on whether $k$ is even or odd.

(ii) Give a formula involving $\text{Res}(f, i/2)$ and $\text{Res}(f, 3i/2)$ for the Fourier transform
\[ \hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x} \, dx \] of \( f \).

**Hint:** Use the identity \( \cosh(z+w) = \cosh(z)\cosh(w) - \sinh(z)\sinh(w) \) to "control" \( f \) as \( R \to \infty \) along the sides of the rectangular contour \( \Gamma_R \) with successive vertices \( R, R+2i, -R+2i, -R \).

Extra Credit (to be attempted only if you have time remaining after completing the rest of the exam) Use more identities for hyperbolic functions to convert your formula in (ii) to the statement that \( f \) is its own Fourier transform.

**Solution.**

(i) The zeros of \( \cosh(\pi z) \) are the points where \( e^{2\pi z} + 1 = 0 \), namely the points called \( z_k \) above. Then
\[
(\frac{d}{dz})_{z=z_k} = \pi \sinh(\pi z_k) = i\pi \sin(\pi k + \pi/2) = (-1)^k \pi i
\]
\( \neq 0 \) implies that each \( z_k \) is a simple pole for \( f \) with
\[
\text{Res}(f, z_k) = \frac{(-1)^k}{i\pi}.
\]

(ii) By the residue theorem, for \( \xi \) fixed and \( g(z) = f(z)e^{-2\pi i \xi z} \), \( i/2 \) and \( 3i/2 \) are the two poles of \( g \) in the interior of \( \Gamma_R \) so
\[
\int_{\Gamma_R} g \, dz = 2\pi i \{ \text{Res}(g, i/2) + \text{Res}(g, 3i/2) \}
\]
\[
= 2 \left( e^{\pi i \xi} - e^{3\pi i \xi} \right)
\]
Using the addition formula for \( \cosh \), it follows that on the two vertical sides of \( \Gamma_R \), \( |g(z)| \leq e^{4\pi |\xi|} / \sinh(\pi R) \) and this quantity goes to 0 as \( R \to \infty \). Also, \( g(x+2i) = e^{4\pi \xi} g(x) \) since
\[
\cosh(\pi x + 2\pi i) = \cosh(\pi x) \cos(2\pi) - i\sinh(\pi x)\sin(2\pi)
\]
\[
= \cosh(\pi x).
\]
It follows that \( \lim_{R \to \infty} \int_{\Gamma_R} g \, dz \)
For $\xi \neq 0$, we obtain
\[
\hat{f}(\xi) = \frac{2e^{\pi\xi}}{1 + e^{2\pi\xi}} = \frac{1}{\cosh(\pi\xi)} f(\xi)
\]
and a continuity argument (or a separate calculation) yields the same conclusion for $\xi = 0$.

II(4) For $z_0$ a point in $\mathbb{C}$, calculate the unique point $w_0$ for which $d_{sph}(z_0, w_0) = 2$ (thus $z_0$ and $w_0$ correspond via stereographic projection to antipodal points on the Riemann sphere $S^2$).

**Solution.** By a routine calculation, with $P_+(w, t) = \frac{2w}{1+t}$ the stereographic projection relative to the south pole $(0, -1)$ from $S^2$ onto $\mathbb{C}$,
\[
w_0 = P_+\left(- (P_+)^{-1}(z_0)\right) = \frac{-z_0}{|z_0|^2} = -1/z_0.
\]

II(5) Compute the value of $\int_0^\infty \frac{\sqrt{x}}{x^2 + 1} \, dx$ by using a keyhole contour in $\mathbb{C}$ and by taking $\sqrt{r e^{i\theta}}$ to be $\sqrt{r} e^{i\theta/2}$ for $r > 0$, $\theta \in (0, 2\pi)$.

**Solution.** Using the stated definition of $\sqrt{z}$, let
\[
f(z) = \frac{\sqrt{z}}{z^2 + 1}. \text{ Then } f \text{ is positive on } [0, \infty), \text{ holomorphic on } \mathbb{C} \setminus [0, \infty) \text{ with poles at } \pm i, \text{ and, for } x > 0,
\]
\[
limit_{y \to 0^+} f(x + iy) = f(x) = - \limit_{y \to 0^+} f(x - iy). \text{ For}
\]
$0 < r < 1 < R$, let $\Gamma_{r,R}$ be the simple closed keyhole contour consisting of a horizontal line segment slightly above the $x - axis$, a counterclockwise arc $C_R$ of radius $R$ about 0, a horizontal line segment slightly below the $x - axis$, and a clockwise arc $C_r$ of radius $r$ about 0. Since

$$|\int_{C_r + C_R} f \, dz| < \frac{2\pi r^{3/2}}{1 - r^2} + \frac{2\pi R^{3/2}}{R^2 - 1}$$

and the right hand side goes to 0 as $r \to 0$ and $R \to \infty$,

$$\int_0^\infty f(x) \, dx = \frac{1}{2} \lim_{r \to 0} \lim_{R \to \infty} \int_{\Gamma_{r,R}} f \, dz$$

$$= \pi i (\text{Res}(f, i) + \text{Res}(f, -i))$$

$$= \pi/2 \left( e^{i\pi/4} - e^{i3\pi/4} \right)$$

$$= \pi \cos(\pi/4)$$

$$= \frac{\pi \sqrt{2}}{2}$$

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II(6) (i) Calculate the disk $D$ of convergence for the power series $\sum_{n \geq 1} \frac{2^n}{n^2} (z - 1)^{4n}$. $\text{Hint: Use a substitution.}$

(ii) Let $f$ be the holomorphic function on $D$ expressed by the power series in (i). Does $f$ have a continuous extension to $\overline{D}$?

Extra Credit (as in II(3), don't pursue this unless you've completed all of the other problems). Does the function $f$ extend to a holomorphic function on a domain $E$ containing $\overline{D}$? If so, how large can $E$ be?

Solution. (i) It's obvious that the power series
\[ \sum_{n \geq 1} \frac{w^n}{n^2} \] has radius of convergence 1 and hence describes a holomorphic function \( g(w) \) on \( \mathbb{D} \). The substitution \( w = 2(z - 1)^4 \) tells us that \[ \sum_{n \geq 1} \frac{2^n}{n^2} (z - 1)^{4n} \] has \( \mathbb{D} = \mathbb{D}(1, 2^{-1/4}) \) as its disk of convergence and hence describes a holomorphic function \( f(z) \) on \( \mathbb{D} \).

\[ (iii) \] Since \[ \sum_{n \geq 1} \frac{1}{n^2} < \infty \], the power series for \( g \) converges absolutely and uniformly to a continuous function on \( \overline{\mathbb{D}} \) and it follows that the power series for \( f \) converges absolutely and uniformly to a continuous function on \( \overline{\mathbb{D}} \).

\[ (iii) \text{ On } \mathbb{D}, \ g'(w) = \frac{1}{w} \sum_{n \geq 1} \frac{w^n}{n} = \frac{\log(1-w)}{w}. \] Let \( G(w) \) be the unique anti-complex derivative of \( \frac{\log(1-w)}{w} \) on \( \mathbb{C} \setminus [1, \infty) \) for which \( G(0) = 0 \). Then \( g \) is the restriction of \( G \) to \( \mathbb{D} \) and \( f \) is the restriction to \( D \) of a holomorphic function \( F \) defined on a quarter-plane.

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II(7) Prove that \( \mathbb{D} = \mathbb{D} \setminus \{0\} \) isn't a Dirichlet domain by showing that there is no member of \( \text{Har}(\overline{\mathbb{D}}) \) which is zero on \( \partial \mathbb{D} \) and has the value 1 at 0.

**Solution.** Suppose \( u \in \text{Har}(\overline{\mathbb{D}}) \). Since harmonic functions on \( \mathbb{D} \) are of the form \( u(z) = b \ln(|z|) \)
\[ + \Re (f(z)) \] for \( f \in \text{Hol}(\mathbb{D}) \) and 
\[ u(0) = \lim_{r \to 0^+} \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta})d\theta \] is finite while 
\[ \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})d\theta \] is the constant term in the Laurent expansion of \( f \) on \( \mathbb{D} \), we 
deduce first that \( b=0 \) and second that \( 0 \) is 
a removable singularity for \( f \), giving \( u \in \text{Har}(\overline{\mathbb{D}}) \). Since 
\( \mathbb{D} \) is a Dirichlet domain, \( u = 0 \) on \( \partial \mathbb{D} \) implies \( u = 0 \) on \( \mathbb{D} \) 
so we can't have \( u(0) = 1 \).

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II(8) Find explicitly the unique harmonic function \( u \) 
on \( \mathbb{D} \) with boundary values \( g(e^{i\theta}) = 4\cos^2 \theta + 2\sin 2\theta \) and 
also find explicitly the unique harmonic function \( v \) on \( \mathbb{D} \) 
for which \( v(0) = 0 \) and \( u + iv \) is holomorphic on \( \mathbb{D} \). 

\textbf{Hint: Use trig identities to calculate the Fourier coefficients of} \( g \).

\textbf{Solution.} From \( g(e^{i\theta}) = 4 (\frac{e^{i\theta} + e^{-i\theta}}{2})^2 + 2 (\frac{e^{i\theta} - e^{-i\theta}}{2i}) \) 
\[ = e^{2i\theta} - ie^{i\theta} + 2 + ie^{-i\theta} + e^{-2i\theta}, \]
we read off that \( \hat{g}(k) \) is 0 for \( |k| > 2 \), \( \hat{g}(0) = 2 \), 
\( \hat{g}(-1) = i = -\hat{g}(1) \), and \( \hat{g}(2) = 1 = \hat{g}(-2) \). 
Substitution of these values into the general formulas 
\[ u(re^{i\theta}) = \sum_{k \in \mathbb{Z}} \hat{g}(k) r^{|k|} e^{ik\theta} \]
\[ v(re^{i\theta}) = \frac{1}{i} \sum_{k \in \mathbb{Z}} \hat{g}(k) \text{sgn}(k) r^{|k|} e^{ik\theta} \]
provides the desired functions \( u \) and \( v \) on \( \mathbb{D} \).

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II(9) Suppose $F$ is a one-to-one conformal map from a domain $D \subset \mathbb{C}$ onto $\mathbb{D}$ which extends to a continuous 1-1 map from $\overline{D}$ onto $\overline{\mathbb{D}}$. Recall from the last homework assignment that $O^2(D)$ is the space of holomorphic functions on $D$ which are square integrable with respect to Lebesgue measure on $D$.

(i) Use the $\mathbb{R}^2$ change of variable theorem for the substitution $z = F(\zeta)$ to show that $(\Phi g)(\zeta) = F'(\zeta)$ $(g \circ F)(\zeta)$ defines a linear isometry $\Phi$ from $O^2(\mathbb{D})$ onto $O^2(D)$.

(ii) Recall from the last homework assignment that $K_D(z, w) = \frac{1}{\pi(1-\overline{z}w)}$ is the reproducing kernel for $O^2(D)$. Use this and (i) to compute $K_D(F^{-1}(z), F^{-1}(w))$.

Solution. (i) Since $F'$ is holomorphic and non-zero on $D$, 

$f(\zeta) = (\Phi g)(\zeta) = (F'(\zeta))(g \circ F)(\zeta)$ defines a 1-1 linear map $f = \Phi g$ from $\text{Hol}(\mathbb{D})$ onto $\text{Hol}(D)$. Let 

$\lambda = \lambda_2$ be Lebesgue measure on $\mathbb{R}^2 = \mathbb{C}$. Since $|F'(\zeta)|^2$ is the determinant of the Jacobian matrix of $(dF)_\zeta$ at $\zeta$, the $\mathbb{R}^2$ change of variable theorem tells us that 

$(d\lambda)(F(\zeta)) = |F'(\zeta)|^2 d\lambda(\zeta)$ and, via the substitution 

$z = F(\zeta)$, $\int_D |g(z)|^2 d\lambda(\zeta) = \int_D |(\Phi g)(\zeta)|^2 d\lambda(\zeta)$ 

for each $g \in \text{Hol}(\mathbb{D})$. Hence $\Phi$ is a linear isometry from $O^2(\mathbb{D})$ onto $O^2(D)$.

(ii) Let $z, w$ be in $\mathbb{D}$ with $\zeta, \eta$ the points in $D$ for which $z = F(\zeta)$, $w = F(\eta)$. For each $f = \Phi g$ in $O^2(D)$,
we have
\[ f(\zeta) = F'(\zeta)g(z) = \int_{D} F'(\zeta)g(w) K_{D}(z, w) d\lambda(w) \]
\[ = \int_{D} F'(\zeta)g(F(\eta)K_{D}(F(\zeta), F(\eta))|F'(\eta)|^{2}d\lambda(\eta) \]
\[ = \int_{D} f(\eta) F'(\zeta) K_{D}(F(\zeta), F(\eta)F'(\eta) d\lambda(\eta) \]
from which we read off that
\[ K_{D}(\zeta, \eta) = \frac{F'(\zeta)K_{D}(F(\zeta), F(\eta)F'(\eta)}{\pi(1-F(\zeta)F(\eta))} \]
As expected, this is a function holomorphic in \( \eta \) and anti-holomorphic in \( \zeta \).