DO THE FOLLOWING PROBLEMS:

1. For $N \geq 2$, suppose $u$ is a twice continuously differentiable function on a domain $D \subset \mathbb{R}^N$.

   $(i)$ Use the second derivative test in one-variable calculus to deduce that, if $\nabla^2 u > 0$ at each point in $D$, $u$ has no maximum value on $D$ while, if $\nabla^2 u < 0$ at each point in $D$, $u$ has no minimum value on $D$.

   $(ii)$ If $u$ is harmonic on $D$ and $D$ is a bounded domain, apply $(i)$ to the functions $u_\varepsilon(x) = u(x) + \varepsilon \lvert x \rvert^2$ and $u_{-\varepsilon}(x) = u(x) - \varepsilon \lvert x \rvert^2$ for each $\varepsilon > 0$ to deduce that $u$ is either constant on $D$ or has no max or min values on $D$.

   $(iii)$ Can you extend the result in $(ii)$ to unbounded domains? [Don't waste a lot of time with this if you have trouble with it]

2. In differential geometry/mathematical physics, when a smooth surface $S$ is the boundary of a simply connected bounded domain $B \subset \mathbb{R}^N$ and $n$ is the outer unit normal field on $S$ (thus, for each $x$ in $S$, $n(x)$ is perpendicular to the tangent space of $S$ at $x$ and points away from $B$), it's customary to write $\partial u/\partial n(x)$ for the directional derivative at $x \in S$ in the direction $n(x)$ of a continuously differentiable function $u$ on some domain containing $S \cup B$. Thus, with $\nabla u(x) = \nabla u(x)$ the gradient field of $u$ and $\langle , \rangle$ the Euclidean inner product
(dot product), $\partial u/\partial n(x) = \langle \nabla u(x), n(x) \rangle$. By the divergence theorem (also called Gauss's Theorem), when $u$ is twice continuously differentiable, $d\sigma(x)$ is the increment of $(N-1)$-dimensional surface volume (arc length for $N=2$, area for $N=3$) on $S$ and $d\text{vol}(x)$ is the increment of $N$-dimensional volume on $B$, we have

$$\int_S \partial u / \partial n(x) \, d\sigma(x) = \int_B \nabla^2 u(x) \, d\text{vol}(x) \quad (1)$$

Use equation (1) to deduce the following for $u$ and $D$ as in Problem 1:

$(i)$ When $\nabla^2 u \geq 0$ on $D$, then $u$ is subharmonic in the sense that $u(x_0) \leq$ average value of $u$ on the boundary of any open ball in $D$ centered at $x_0$;

$(ii)$ When $\nabla^2 u \leq 0$ on $D$, then $u$ is superharmonic in the sense that $u(x_0) \geq$ average value of $u$ on the boundary of any open ball in $D$ centered at $x_0$;

$(iii)$ If $u$ is harmonic on $D$, then $u$ has the mean value property.

3. In class, we'll show that the solution of the Dirichlet problem for $(\mathbb{D}, g^\sim)$ is given by the Poisson integral

$$u^\sim(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g^\sim(e^{i\phi}) P_r(\theta - \phi) \, d\phi \quad (1)$$

where the Poisson kernel $P_r(\theta)$ for $\mathbb{D}$ is defined to be the real part of $\frac{1 + re^{i\theta}}{1 - re^{i\theta}}$. Now suppose we start with a continuous (or perhaps just integrable) function $g$ on $\partial \mathbb{H} = \mathbb{R} \cup \{\infty\}$ and seek the solution $u$ of the Dirichlet problem for $(\mathbb{H}, g)$. Since the Cayley transform $T(z) = \frac{z-i}{z+i}$
maps $\mathbb{H}$ conformally and 1-1 onto $\mathbb{D}$ and also maps $\partial \mathbb{H}$ homeomorphically onto $\partial \mathbb{D}$, putting $g^\sim = g \circ T^{-1}$, it's immediate that $u = u^\sim \circ T$, where $u^\sim$ solves the Dirichlet problem for $(\mathbb{D}, g^\sim)$. On the other hand, we'll show in class that
\[
u(x + iy) = \int_{\mathbb{R}} g(t)P_y(x-t)
\]
where, as in a previous homework problem, the Poisson kernel $P_y(x)$ for $\mathbb{H}$ is defined to be $\frac{1}{\pi} \left( \frac{y}{x^2 + y^2} \right) = \text{real part of} \ \frac{i}{\pi(x+iy)}$. Check that these two ways of describing $u$ are consistent by transforming (1) into (2) by the substitutions $re^{i\theta} = T(x + iy)$, $e^{i\phi} = T(t)$). [You'll need to "invent" some tricks to compute $d\phi/dt$ and will need to "keep your eyes open" with the algebra associated with the substitutions].

3. In practice, when we want to solve the Dirichlet problem explicitly for some "reasonably nice" simply connected domain $D$ and every $g \in C(\partial D, \mathbb{R})$, we try to construct an explicit 1-1 conformal mapping from $D$ onto $\mathbb{D}$ or from $D$ onto $\mathbb{H}$, whichever is easier, then apply (1) or (2) and transform back to wind up with an integral of $g$ times a kernel function for $D$. Carry this out for the following domains:

(i) The sector $\{z \in \mathbb{C} : 0 < \text{Arg}(z) < \alpha\}$ where $\alpha < 2\pi$;
(ii) The vertical strip $\{z \in \mathbb{C} : 0 < \text{Re}(z) < 1\}$;
(iii) $\mathbb{H} \cap \mathbb{D}$ where we limit attention to boundary functions $g$ which satisfy $g(x) = g(-x)$ for $x \in [-1, 1]$;
(iv) $\mathbb{H} \cap \mathbb{D}$ for general boundary functions.
4. The Neumann problem for a bounded domain $D$ having a smooth boundary and a boundary value function $h$ on $\partial D$ is to find a harmonic function $u$ on $D$ for which, in "some reasonable sense", $\partial u / \partial n = h$ on $\partial D$. Solve the Neumann problem for $D=\mathbb{D}$ by relating it to the Dirichlet problem in a suitable way. Is there an associated kernel function for solutions of the Neumann problem?