ASSIGNMENT #4 (DUE MONDAY, SEPTEMBER 28)

1. #14, p. 55

2. #16, p. 55

3. #18, p. 55

4. Let $D$ be the domain in $\mathbb{C}$ bounded by two parallel lines. Construct a 1-1 conformal map $f$ from $D$ conformally onto the standard unit disc $\mathbb{D}$.

   *Hint.* As we'll discuss in class, the Cayley transform $z \mapsto \frac{z-i}{z+i}$ maps the upper half plane $\mathbb{H}^+ = \{z = x + iy : y > 0\}$ conformally onto $\mathbb{D}$. Construct $f$ as the composition of a suitable Möbius transformation followed by the exponential map followed by the Cayley transform.

5. Consider distinct points $z_0$ and $z_0^*$ and the 6 domains in $\mathbb{C}$ determined by three distinct circles in $\overline{\mathbb{C}}$ each passing through $z_0$ and $z_0^*$. Pick a typical one of these domains and describe a 1-1 conformal map from your chosen domain onto $\mathbb{D}$. Be sure to indicate how your map depends on the angle between its boundary curves. As in #2, you'll probably want to use a composition of a Möbius transformation, a power function, and the Cayley transform.
6. #19, p.56

7. Let \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C}) \) with \( g \neq \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), \( \tau = (a + d)/2 \), and \( T(z) = T_g(z) = \frac{az+b}{cz+d} \). Note that since \( \det g = 1 \), \( g \) either has two distinct eigenvalues whose product is 1 or \( g \) is not diagonalizable.

\((i)\) Show that \((z_1, z_2) \in \mathbb{C}^2\) is a non-zero eigenvector for the linear transformation on \( \mathbb{C}^2 \) with matrix \( g \)
\( \iff T(z_1/z_2) = z_1/z_2 \).

\((ii)\) Show that \( g \) is parabolic in the sense that \( g \)
has only one eigenvalue
\( \iff \tau = \pm 1 \)
\( \iff g \) is similar to \( \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \)
\( \iff T \) has only one fixed point \( z_0 \)
and deduce that, for each \( z \in \overline{\mathbb{C}} \), \( \lim_{|k| \to \infty} T^k z = z_0 \).

\((iii)\) Show that \( g \) is hyperbolic in the sense that \( g \)
has two distinct eigenvalues \( \lambda \) and \( 1/\lambda \) with \( |\lambda| < 1 \)
\( \iff \tau \notin [-1, 1] \)
\( \iff T \) has two distinct fixed points \( z_0 \) and \( z_0^* \) with
\( \lim_{k \to \infty} T^k z = z_0, \lim_{k \to \infty} T^{-k} z = z_0^* \) for each \( z \in \overline{\mathbb{C}} \backslash \{z_0, z_0^*\} \).

\((iv)\) Show that \( g \) is elliptic in the sense that \( g \) has two distinct eigenvalues each of magnitude 1
\[ -1 < \tau < 1 \]
\[ \Leftrightarrow \text{T has two distinct fixed points } z_0 \text{ and } z_0^* \text{ for which } \lim_{k \to \infty} T^k z \text{ nor } \lim_{k \to \infty} T^{-k} z \text{ exists for any } z \in \overline{\mathbb{C}} \setminus \{ z_0, z_0^* \}. \]

Also show that for each such \( z \) with \( C_z \) the circle containing \( z \) and having \( z_0 \) and \( z_0^* \) as symmetric points, either the \( T \)-orbit \( \{ T^k z : k \in \mathbb{Z} \} \) of \( z \) is a finite subset of \( C_z \) or a dense subset of \( C_z \).

What are the conditions on the eigenvalues of \( g \) distinguishing the finite orbit case from the dense orbit case?

8. For \( g = \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix} \), we have \( \tau = 5/4 > 1 \) so \( g \) is hyperbolic. Calculate the attractive fixed point \( z_0 \) for \( g \) and the repelling fixed point \( z_0^* \) for \( T=T_g \). Diagonalize \( g \) in order to get easy formulas for the integer powers of \( T \) and use these to estimate, for \( z \notin \{ z_0, z_0^* \} \) the minimal \( k_0(z) \) for which both \( |T^k z - z_0| \) and \( |T^{-k} z - z_0^*| \) are < \( \frac{1}{1000} \) \( \forall k \geq k_0(z) \).