ASSIGNMENT #2 Due Friday, February 12. Do the following problems:

1. Compute the Green's function \( g(z', z) = g_z(z') \) for the upper half plane \( \mathbb{H} \) by using the formula derived in class for the Green's function on \( \mathbb{D} \) and the Cayley transform \( w = \frac{z-i}{z+i} \) from \( \mathbb{H} \) onto \( \mathbb{D} \). Then check that, with \( P_y(x) = \frac{y}{\pi(x^2+y^2)} \) the Poisson kernel on \( \mathbb{H} \) (as discussed last semester),

\[
\frac{1}{2\pi} \left( \frac{\partial}{\partial s} \right)_{s=0} g(t+is, x+iy) = P_y(x-t).
\]

Note that the derivative on the left hand side is the negative of the outer normal derivative of \( g_{x+iy} \) at the boundary point \( t \in \mathbb{R} \) for \( \mathbb{H} \).

2. As usual, when \( 0 < R_1 < R_2 \), \( A(z_0, R_1, R_2) \) is the annulus \( \{ z \in \mathbb{C} : R_1 < |z-z_0| < R_2 \} \) and it's obvious that this annulus is conformally equivalent to \( A(0, R_1/R_2, 1) \) via the conformal map \( z \mapsto (z-z_0)/R_2 \). It's then convenient to restrict attention to the family of standard annuli \( A(r) = A(0, r, 1), 0 < r < 1. \)

(i) Use a version of Schwarz's Lemma to deduce that \( A(r_1) \) and \( A(r_2) \) are conformally equivalent \( \iff r_1 = r_2 \). It follows immediately that two non-standard annuli \( A(z_0, R_1, R_2) \) and \( A(z_0', R_1', R_2') \) are conformally equivalent \( \iff R_1 / R_2 = R_1' / R_2' \).

(ii) For \( r \in (0, 1) \), compute the group of all 1-1 conformal maps from \( A(r) \) onto itself. \( Hint: \) Note that \( z \mapsto r/z \) is a member of this group.
3. The Koebe 1/4 function is defined by \( k(z) = \frac{z}{(1+z)^2} \).

\((i)\) Use the obvious fact that \( k(z) = k(1/z) \) for \( z \neq 0 \) to deduce that the restrictions of \( k \) to \( \mathbb{D} \) and to \( \mathbb{C} \setminus \mathbb{D} \) are 1-1 conformal maps onto a common simply connected domain \( \mathbb{D} \).

\((ii)\) Show that \( \mathbb{D} \) contains \( \overline{D(0,1/4 \setminus \{1/4\})} \). In particular, this means that \( 1/4 \) is the radius of the largest open disk about 0 contained in \( k(\mathbb{D}) \).

\((iii)\) Show that \( \mathbb{D} \) is the slit domain \( \mathbb{C} \setminus [1/4, \infty) \).

Hence, for the rotated Koebe function \( k_{\theta}(z) \) (see the univalent function notes), \( k_{\theta}(\mathbb{D}) = \mathbb{C} \setminus e^{-i\theta}[1/4, \infty) \) is also a slit domain.

*Read the univalent function notes before embarking on the problems below.*

4. Show that nothing like the Viertel Satz holds for non-univalent functions by consider the holomorphic functions \( f_\varepsilon(z) = \varepsilon(e^{z/\varepsilon} - 1), \varepsilon > 0 \), noting that \( f_\varepsilon(0) = 0 \) and \( f_\varepsilon'(0) = 1 \).

5. Prove Bieberbach's Area Theorem in the following way.

For \( h \in \Sigma \) with the expansion (2), the fact that \( h(0) = \infty \) and \( h \) is univalent means that \( h \) maps \( D(0,r) \) into an unbounded domain whose boundary is the simple closed \( \Gamma(r) = h(C(0,r)) \)

and \( \theta \mapsto h(re^{i\theta}) \) defines a clockwise orientation of \( \Gamma(r) \).

By Green's Theorem, we can calculate the area \( A(r) \) of the interior of \( \Gamma(r) \) by describing each \( w \) in \( \mathbb{C} \) by \( w = u + iv \) and integrating \( \alpha = \frac{1}{2}(vdu - udv) \) over \( \Gamma(r) \). But the complex 1-form

\[ \frac{-w}{2i} \frac{dw}{dz} \]

is equal to \( \alpha + i\beta \) where \( \beta = \frac{1}{2}(udu + vdv) \) is exact.

This means that we can compute \( A(r) \) as

\[ \frac{1}{2i} \int_{0}^{2\pi} h(re^{i\theta}) h'(re^{i\theta}) d(re^{i\theta}). \]

Carry out this integral and then take the limit as \( r \to 1 \) to prove the Area Theorem.
6. (i) First check the details of the passage from \( f \in \mathcal{S} \) to \( g \) satisfying (3). Why does the power series for \( \psi \) converge on \( \mathbb{D} \)? Why is \( g \in \mathcal{S} \)?

(ii) Check that, for \( f \) described by (1) and \( h = 1/g \) described by (2), \( c_0 = 0, \ |c_1| = \frac{|a_2|}{2} \) and deduce from the area theorem that \( |a_2| = 2 \iff h(z) = 1/z + e^{2i\theta} z \) for some \( \theta \in \mathbb{R} \) and go on to deduce that this holds \( \iff f = k_0 \).

7. Prove the Viertel Satz using (6) and checking that when \( f \in \mathcal{S} \) and \( w \notin f(\mathbb{D}) \), \( f^\sim(z) = w f(z)/(w - f(z)) \) is also in \( \mathcal{S} \) with the relationship between \( z^2 \) coefficients for \( f \) and \( f^\sim \) being as described in the Notes.