HOMEWORK ASSIGNMENT #6 Due Friday, March 19.

Do the following problems:

1. Show the following properties for the Hilbert space $\mathcal{B}_1 \subset L^2(\mathbb{R})$ of functions of the form $f(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} F(\xi)e^{2\pi i \xi x} d\xi$, $F \in L^2\left([ - \frac{1}{2}, \frac{1}{2}]\right)$, and their extensions to holomorphic functions on $\mathbb{C}$.

   (i) $\mathcal{B}_1$ is a reproducing kernel space with the kernel function $K(x, y) = \sum_{k \in \mathbb{Z}} \text{sinc}(x - k) \text{sinc}(y - k) = \text{sinc}(x - y)$.

   Extend this formula by analytic continuation to $K(z, w) = \sum_{k \in \mathbb{Z}} \overline{\text{sinc}(z - k)} \text{sinc}(w - k) = \text{sinc}(\overline{z} - w) \forall z, w \in \mathbb{C}$.

   (ii) For $f, g \in \mathcal{B}_1$ and $z \in \mathbb{C}$,

   $\sum_{k \in \mathbb{Z}} f(z - k) \overline{g(z - k)} = \langle f, g \rangle = \int_{\mathbb{R}} f(t) \overline{g(t)} \, dt$.

   (iii) $f, g \in \mathcal{B}_1$ and $z \in \mathbb{C}$,

   $\sum_{k \in \mathbb{Z}} f(z - k) g(k) = \sum_{k \in \mathbb{Z}} f(k) g(z - k) = \int_{\mathbb{R}} f(z - t) g(t) \, dt$.

   Hint: As in (i), for (ii) and (iii), first do the case $z = x \in \mathbb{R}$ by taking the Fourier transform on the left and then use analytic continuation.

2. Carefully check that every lattice in $\mathbb{C}$ is complex equivalent to $\Lambda_\tau$ for a unique $\tau$ with $\text{Im}(\tau) > 0$ and $\text{Re}(\tau) \in [0, 1/2)$. 
3. Let Λ be a lattice in C
   
   (i) Show that $\mathbb{T}_{2\Lambda}$ is a compact Riemann surface using as local coordinate maps $\psi_\mathcal{U}(z) = z + \Lambda$ for $\mathcal{U}$ any domain in $\mathbb{C}$ such that no two points of $\mathcal{U}$ differ by a member of $\Lambda$ (e.g. $\mathcal{U}$ the interior of some $\Lambda$ cell).
   
   (ii) Explain how (i) allows identification of the field $\mathbb{F}_{2\Lambda} = \{\text{constant functions}\} \cup \text{Ell}(2\Lambda)$ with the collection of holomorphic functions from $\mathbb{T}_{2\Lambda}$ into $S^2$.

4. (i) Use a Riemann sum argument to show that
   
   \[
   \sum_{(j,k) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(j^2+k^2)^{3/2}} \quad \text{is finite. \textit{Hint: Compare this infinite series with}} \int_{\{(x,y) \in \mathbb{R}^2 : x^2+y^2 > 1\}} \frac{1}{(x^2+y^2)^{3/2}} \, dx \, dy.
   \]
   
   (ii) Use the equivalence of any two norms in $\mathbb{R}^2$ and (i) to show that, for any lattice $\Lambda$ in $\mathbb{C}$,
   
   \[
   \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{|\lambda|^3} < \infty \quad \text{although} \quad \sum_{\lambda \in \Lambda \setminus \{0\}} \frac{1}{|\lambda|^2} = \infty.
   \]

   \textit{Hint: First choose a lattice basis for $\Lambda$.}
   
   (iii) Use (ii) to show that, for any lattice $\Lambda$, the series
   
   \[
   1/z^2 + \sum_{\lambda \in \Lambda \setminus \{0\}} \left\{ \frac{1}{(z-2\lambda)^2} - \frac{1}{(2\lambda)^2} \right\}
   \]
   
   converges uniformly on compact subsets of $\mathbb{C} \setminus 2\Lambda$ to an even meromorphic function $p_{2\Lambda}$ whose set of poles is the lattice $2\Lambda$ and each such pole has order 2.
   
   (iv) Deduce from (iii) that
   
   \[
   p'_{2\Lambda} = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z-2\lambda)^3}
   \]
   
   is an odd meromorphic function having $2\Lambda$ as its set of poles with each pole being of order 3.
   
   (v) By a simple change of summation variable, show that $p'_{2\Lambda}$ is a member of $\text{Ell}(2\Lambda)$ with order 3 and deduce from this that $p_{2\Lambda}$ is member of $\text{Ell}(2\Lambda)$ of order 2. Use the properties of $\text{Ell}(2\Lambda)$ and the fact that $p'_{2\Lambda}$ is an odd function to show that $\Lambda \setminus 2\Lambda$ is the set of zeros for $p'_{2\Lambda}$ and that each such zero is simple.
5. (i) Use 3(iv),(v) to obtain a very easy proof that, for \{E_1, E_2, E_3\} the half-period values for \(p_{2\Lambda}\),
\[
(p_{2\Lambda}')^2 = 4(\ p_{2\Lambda} - E_1 )(\ p_{2\Lambda} - E_2 )(\ p_{2\Lambda} - E_3 ).
\]
Then go on to derive a similar differential equation involving half-period values for any even member of Ell(2\(\Lambda\)).

(ii) Use property (v) in the Notes on Elliptic Functions to show that if \(f\) is an even member of Ell(2\(\Lambda\)) \(\Leftrightarrow f = R_0 \circ p_{2\Lambda}\)

*Hints*: First reduce to the case when \(f\) has no zeros or poles on 2\(\Lambda\). Then use the fact that \(f\) is even to see that \(f(a) = 0 \Leftrightarrow f(-a) = 0\) and go on to construct a rational combination of \(p_{2\Lambda}\) having the same zeros as \(f\).

(iii) Prove the Fundamental Theorem of Elliptic Functions from (ii) and the observation that \(f\) is an odd member of Ell(2\(\Lambda\)) \(\Leftrightarrow f / p_{2\Lambda}'\) is an even member of Ell(2\(\Lambda\))