1. As mentioned in class, Blaschke products are products, with convergence factors, of Möbius transformations of the form $T_a(z) = \frac{|a|}{a} \frac{z - a}{az - 1}$ with $a \in \mathbb{D} \setminus \{0\}$ and, although finite Blaschke products have nice properties (e.g., extend continuously to maps from $\mathbb{D}$ onto $\overline{\mathbb{D}}$, infinite Blaschke products are examples of holomorphic functions from $\mathbb{D}$ into $\mathbb{D}$ with bad properties (e.g., never extend continuously to $\partial \mathbb{D}$)

(i) The reason for including the rotation factor $\frac{|a|}{a}$ in $T_a$ is because it leads to a nice estimate for $| 1 - T_a(z) |$. Show that, for each $z \in \mathbb{D}$,

$$| 1 - T_a(z) | = (1 - |a|) \frac{|z - a|/|a|}{|1 - az|} \leq (1 - |a|) \frac{1 + |z|}{1 - |z|}$$

(ii) Let $Z/(a_k)_{k \in \mathbb{N}}$ be a sequence in $\mathbb{D} \setminus \{0\}$ for which $\lim_{k \to \infty} |a_k| = 1$. Use (i) to show that we can choose sequences $N/(n_k)_{k \in \mathbb{N}}$ of non-negative integers for which $\prod_{k=1}^\infty E_k (1 - T_{a_k}(z))$ converges unconditionally and uniformly on compact subsets of $\mathbb{D}$ to a holomorphic function $B_{Z/N}$ mapping $\mathbb{D}$ into $\mathbb{D}$. When $\sum_{k \in \mathbb{N}} (1 - |a_k|)^s < \infty$ for some $s > 0$, explain why we can use the constant sequence $n_k = n$ with $n$ the smallest non-negative integer for which $\sum_{k \in \mathbb{N}} (1 - |a_k|)^{n+1} < \infty$ to obtain a canonical Blaschke product $B_{Z,n}$. 
Definitions. When $Z=(a_k)_{k\in\mathbb{N}}$ is a sequence of non-zero complex numbers for which $\lim_{k \to \infty} |a_k| = \infty$, then

$S_Z = \{ s > 0 : \sum_{k \in \mathbb{N}} \frac{1}{|a_k|^s} < \infty \}$ defines the set of convergence exponents for $Z$. When $S_Z \neq \emptyset$, $\sigma_z = \inf S_Z$ is called the critical exponent for $Z$ and we then have $\sum_{k \in \mathbb{N}} \frac{1}{|a_k|^s} < \infty$.

$\forall s > \sigma_z$ and $\sum_{k \in \mathbb{N}} \frac{1}{|a_k|^s} = \infty \forall s' < \sigma_z$. However, $\sigma_z$ may or may not be in $S_Z$. When $S_Z = \emptyset$, it's convenient to put $\sigma_z = \infty$.

For example, when $a_k = k$, $S_Z = (1, \infty)$ and $\sigma_z = 1 \notin S_Z$. On the other hand, for $a_k = k(\ln(k))^2$, $S_Z = [1, \infty)$ and $\sigma_z = 1 \in S_Z$.

2. Show that, for each canonical product $E=E_{Z,n}$, $\lambda_E \leq s$ $\forall s \in S_Z$ and hence $\lambda_E \leq \sigma_z$.

Remarks. We showed in class (part of our proof of Hadamard-Part I) that, when $S_Z \neq \infty$ and $n$ is the smallest non-negative integer for which $n+1 \in S_Z$, then the canonical product

$E_{Z,n} = \prod_{k \in \mathbb{N}} E_n(z/a_k)$ has growth order $\leq n+1$. Hadamard-Part II (soon to be proved in class) says that if $Z$ is the sequence of non-zeros of some $f \in \text{Hol}_{FG}(\mathbb{C})$, then each $\lambda > \lambda_f$ lies in $S_Z$.

In particular, for each canonical product $E=E_{Z,n}$, we have $n \leq \sigma_z \leq \lambda_E \leq n+1$. By Problem 2, we then have $\sigma_z = \lambda_E$ and the convergence exponent set $S_Z$ coincides with the order exponent set $O_E$ defined in class. If $\sigma_z \notin \mathbb{Z}^+$, there is no ambiguity since we necessarily have $n < \sigma_z = \lambda_E < n + 1$. But, when $\sigma_z = m$ is an integer, the ambiguity about $\lambda_E$ is resolved as follows:

(i) If $m \notin S_Z$, then $m = \lambda_E = n$. 
(ii) If \( m \in S_Z \), then \( m = \lambda_E = n + 1 \).

3. We know that there is a 1-1 correspondence between sequences \( c = (c_k)_{k \geq 0} \) for which \( \limsup_{k \to \infty} |c_k|^{1/k} = 0 \) and holomorphic functions \( f_c \) on \( \mathbb{C} \) given by
\[
f_c(z) = \sum_{k=0}^{\infty} c_k z^k.
\]
This leads to exploring the relationship between \( c \) and \( \lambda_{f_c} \).

(i) Fix \( c \) as above and let \( f = f_c \). Show that
\[
\mathcal{O}_f = \{ \lambda > 0 : A_{\lambda} = \sup_{\mathbb{C}} |f(z)|/|z|^\lambda \} < \infty\}
\]
with \( \mathcal{R}_c = \{ \mu > 0 : B_{\mu} = \sup_{\mathbb{Z}^+} |c_k|^{1/k} \} \). It follows that \( f \in \text{Hol}_F(G) \) \( \Leftrightarrow \mathcal{R}_c \neq \emptyset \) and then \( \lambda_f \) coincides with \( \mu_c = \inf(\mathcal{R}_c) \). By an easy argument analogous to that in the proof of Lemma 7.5.2, \( \mu_c = \limsup_{k \to \infty} \frac{\ln(k)}{\ln(|c_k|^{1/k})} \) and hence
\[
\alpha_c = \frac{1}{\mu_c} = \liminf_{k \to \infty} \frac{-\ln(|c_k|)}{k \ln(k)}.
\]
The steps outlined by Conway in Problem 5, Section XI.2, p.288, provide a guide to the non-trivial estimations needed to show that \( \mathcal{O}_f = \mathcal{R}_c \).

(ii) Use the result in (i) to prove that, for \( \rho > 0 \),

the function \( f(z) = \sum_{k=0}^{\infty} \frac{z^k}{(k!)^{1/\rho}} \) has growth order \( \rho \).

4. Use the Hadamard Theorem to prove, by contradiction, that the equation \( e^z = z \) has infinitely many solutions. More generally, if \( p \) and \( q \) are non-zero polynomials, show that the equation \( e^{p(z)} = q(z) \) has infinitely many solutions.

5. Show that, for \( \alpha > 1 \),
\[
f_{\alpha}(z) = \int_{-\infty}^{\infty} e^{-|t|^\alpha} e^{2\pi i z t} \, dt
\]
is holomorphic on \( \mathbb{C} \) with growth order \( \alpha/(\alpha - 1) \).

\textbf{Hint: } Find a constant \( C \) for which
\[-|t|^{\alpha/2} + 2\pi |z| |t| \leq C |z|^{\alpha/(\alpha-1)}\]

6. Show that, for $\text{Im}(\tau) > 0$, the Jacobi theta function

\[z \mapsto \Theta(z, \tau) = \sum_{k=-\infty}^{\infty} e^{\pi i (k^2 \tau + 2kz)}\]

is holomorphic on $\mathbb{C}$ with growth order 2.

Hint: Check that, for $t > 0$ and $k \geq 4 |z|/|t|$, 
\[-k^2 t + 2k |z| \leq k^2 t/2.\]

7. Compute the Hadamard factorizations for the functions 

\[f(z) = e^z - 1\] and, for $c \neq 0$, 
\[f_c(z) = f(z - c).\]

8. Fix $t > 0$ and let $f(z) = \prod_{k=1}^{\infty} (1 - e^{2\pi i (z - kt)})$

(i) Show that $f$ is holomorphic on $\mathbb{C}$ with \(\{m - int : n \in \mathbb{N}, m \in \mathbb{Z}\}\) the set of zeros of $f$. Then use Problem 2 to obtain an easy proof that $f$ has growth order $\geq 2$: thus, with $\mathcal{Z}$ some sequence iterating the non-zero zeros of $f$, check that $S_\mathcal{Z} = (2, \infty)$ by the usual device of comparing sums over $\mathcal{Z}^2 \setminus \{0\}$ with integrals over $\mathbb{R}^2$.

(ii) For any $N$, we can write $f = f_N g_N$, where 
\[f_N(z) = \prod_{k=1}^{N} (1 - e^{2\pi i (z - kt)}) \quad \text{and} \quad g_N(z) = \prod_{k=N+1}^{\infty} (1 - e^{2\pi i (iz - kt)}).\]

Show that, if $c > 0$ is sufficiently large and $N(z)$ is the largest integer $\leq c |z|$, then $|g_{N(z)}(z)| \leq A$ where $A$ doesn't depend on $z$ and $|f_{N(z)}(z)| \leq \prod_{k=1}^{N(z)} (1 + e^{2\pi |z|}) \leq e^{B|z|^2}$ for an appropriate choice of $B$.

(iii) Deduce from (i) and (ii) that $\lambda_f = 2$. By Hadamard's Theorem, $f$ has the standard factorization 
\[f = E_{\mathcal{Z}, 2} e^g\]
with \( g \) a polynomial of degree \( \leq 2 \). Despite the fact
that Problem 7 provides a "first step" toward obtaining
\( g \), don't try to grind out the computations needed to explicitly
determine \( g \).

\textit{Recall:} When \( a_0, a_1, \ldots, a_N \) is a finite sequence of distinct
complex numbers, the \( j^{th} \) Lagrange interpolation polynomial
for the sequence is defined by
\[
\phi_j(z) = \prod_{0 \leq i \neq j \leq N} \frac{(z-a_i)}{(a_j-a_i)}
\]
and it follows that, for \( c_0, c_1, \ldots, c_N \) in \( \mathbb{C} \), Lagrange's
interpolation formula
\[
p(z) = \sum_{i=0}^{N} c_j \phi_j(z)
\]
defines the unique polynomial \( p \) of degree \( \leq N \) for which \( p(a_k) = c_k \) for \( 0 \leq k \leq N \).

9. (Pringsheim's Interpolation formula for \( \text{Hol}(\mathbb{C}) \)). Let
\( \mathcal{Z} = (a_k)_{k \in \mathbb{N}} \) be a sequence in \( \mathbb{C} \setminus \{0\} \) with \( \lim_{k \to \infty} |a_k| = \infty \)
and let \( \mathcal{N} \) be a sequence of distinct non-negative integers for
which we have a regular/Weierstrass infinite product \( E_{\mathcal{Z}, \mathcal{N}} \). For
each \( j \in \mathbb{N} \), let \( \phi_j(z) = \frac{E_{\mathcal{Z}, \mathcal{N}}(z)}{E_{\mathcal{Z}, \mathcal{N}}(a_j)(z-a_j)} \). Clearly, \( \phi_j \in \text{Hol}(\mathbb{C}) \)
with \( \phi_j(a_k) = 0 \) for \( k \neq j \) and \( \phi_j(a_j) = 1 \).

(i) Show that, for each sequence \( \mathcal{V} = (c_k)_{k \in \mathbb{N}} \) in \( \mathbb{C} \),
we can choose a sequence \( \mathcal{M} = (m_k)_{k \in \mathbb{N}} \) of non-negative integers
for which \( \sum_{j \in \mathbb{N}} c_j \phi_j(z)(z/a_j)^{m_j} \) converges uniformly on compact
subsets of \( \mathbb{C} \) to \( f = f_{\mathcal{V}} \in \text{Hol}(\mathbb{C}) \) with \( f(a_k) = c_k \ \forall k \in \mathbb{N} \).

(ii) Exhibit a similar interpolation formula when \( \mathcal{Z} \) is
replaced by \( \mathcal{Z} \cup \{0\} \) and \( \mathcal{V} \) is replaced by \( \mathcal{V} \cup \{c_0\} \).
(iii) When $\sigma_z < \infty$ and we replace $E_{Z,N}$ with the canonical product $E_{Z,n}$, are the holomorphic functions $f_Y$ in $\text{Hol}_{FG}(\mathbb{C})$?

Remarks. (a) The functions $f_Y$ defined by Pringsheim’s Interpolation Formula are not unique: indeed, using the Weierstrass Theorem, $h$ is a holomorphic function with $h(a_k) = c_k \forall k \in \mathbb{N} \iff h = f_Y + E_{Z,N}e^g$ for some $g \in \text{Hol}(\mathbb{C})$ and, even in the case when $\sigma_z < \infty$, we don’t get a useful uniqueness statement by imposing limitations on the growth order of $h$.

(b) On the other hand, when $\sigma_z = \infty$, Hadamard’s Theorem implies that $E_{Z,N}e^g$ has infinite growth order for every $g$ and it follows that $Z$ is a set of uniqueness for $\text{Hol}_{FG}(\mathbb{C})$ in the sense that, for each choice of a value sequence $V$, there is at most one $f \in \text{Hol}_{FG}(\mathbb{C})$ for which $f(a_k) = c_k \forall k \in \mathbb{N}$; equivalently, for $f_1$ and $f_2$ in $\text{Hol}_{FG}(\mathbb{C})$, $f_1(z) = f_2(z) \forall z \in \mathbb{C} \iff f_1(a_k) = f_2(a_k) \forall k \in \mathbb{N}$. For example, $(\ln k)_{k \in \mathbb{N}}$ and all of its subsequences are sets of uniqueness for $\text{Hol}_{FG}(\mathbb{C})$.

(c) The space $\text{Hol}_{FG}(\mathbb{C})$ is huge since it is open and dense in $\text{Hol}(\mathbb{C})$ relative to the compact open topology. Intuitively, the sets of uniqueness for $\text{Hol}_{FG}(\mathbb{C})$ are the discrete subsets which "come close" to being non-discrete in the sense that their magnitudes go to infinity very slowly.

When we look at a closed subspace $\mathcal{H}$ of $\text{Hol}(\mathbb{C})$ which, relative to some inner product, is a reproducing kernel space with kernel function $K_z(w)$, any $Z = (a_k)_{k \in \mathbb{N}}$ for which $\{K_{a_k} : k \in \mathbb{N}\}$ spans a dense subspace of $\mathcal{H}$ (equivalently, there is no non-zero $f$ in $\mathcal{H}$ for which $f(a_k) = 0 \forall k \in \mathbb{N}$) is a set of uniqueness for $\mathcal{H}$. For example, $Z$ is a set of uniqueness for $\mathbb{B}_1(\mathbb{C}) \subset \text{Hol}_{FG}(\mathbb{C})$. In applications, function spaces for which there are countable sets of uniqueness are of great interest.