NOTES ON ELLIPTIC FUNCTIONS

Background on Lattices

In an additive, locally compact, separable group \((G,+),\) a lattice in \(G\) is, by definition, a discrete subgroup \(\mathcal{L}\) of \(G\) for which, relative to the quotient topology, the quotient group \(T_\mathcal{L} = \mathcal{G}/\mathcal{L}\) is compact. Then \(f : G \rightarrow \mathbb{C}\) is said to be \(\mathcal{L}\)-periodic if \(f(x + l) = f(x)\) \(\forall(x, l) \in G \times \mathcal{L}\) and the entire theory of Fourier series carries over to the study of locally integrable \(\mathcal{L}\)-periodic functions on \(G\) or, equivalently, integrable functions on \(T_\mathcal{L}\). As a result, much of abelian harmonic analysis reduces via introduction of lattices to Fourier series results.

For the special case of a vector group \((V,+),\) with \(V\) an \(n\)-dimensional vector space over \(\mathbb{R}\), it's not hard to use induction on \(n\) to show that \(\mathcal{L}\) is a lattice in \(V\) if there exists an \(\mathbb{R}\)-basis \(\mathcal{B} = \{v_1, \cdots, v_n\}\) such that

\[
\mathcal{L} = \text{span}_\mathbb{R} \mathcal{B} = \{\sum_{j=1}^{n} k_j v_j : (k_1, \cdots, k_n) \in \mathbb{Z}^n\}
\]

and any such \(\mathcal{B}\) is said to be a basis for \(\mathcal{L}\) with a trivial argument showing that

\[
\sum_{j=1}^{n} x_j v_j + \mathcal{L} \mapsto (x_1, \cdots, x_n) + \mathbb{Z}^n
\]

is both an algebraic isomorphism and a homeomorphism from \(T_\mathcal{L}\) onto the standard \(n\)-torus \(\mathbb{T}^n = \mathbb{R}^n\). Although \(T^n\) and \(T_\mathcal{L}\) can be imbedded in \(\mathbb{C}^n\) as the product of \(n\) copies of the unit circle \(S^1 = \{z \in \mathbb{C} : |z| = 1\}\), it's preferable to imbed \(T_\mathcal{L}\) as a toroidal \(n\)-dimensional surface in \(\mathbb{R}^{n+1}\) by choosing a lattice basis \(\mathcal{B}\), a point \(v_0 \in V\), and considering the \(\mathcal{L}\)-cell \(P = P_{a,B} = \{a + \sum_{j=1}^{n} x_j v_j : (x_1, \cdots, x_n) \in [0,1]^n\}\). Although \(P\) is in 1-1 correspondence with \(T_\mathcal{L} = V/\mathcal{L}\), the quotient topology on \(V/\mathcal{L}\) is tantamount to glueing together in \(\mathbb{R}^{n+1}\) the opposite sides of \(P\).

It's obvious that any two lattices \(\mathcal{L}_1\) and \(\mathcal{L}_2\) in \(V\) are equivalent in the sense that there is an invertible linear transformation \(T : V \hookrightarrow V\) such that \(\mathcal{L}_2 = T(\mathcal{L}_1)\). Also, for \(\mathcal{L}\) a lattice in \(V\) and \(T \in \text{GL}(V)\) = \{invertible linear operators on \(V\)\}, \(T(\mathcal{L}) = \mathcal{L} \mapsto T(\mathcal{B})\) is a lattice basis for \(\mathcal{L}\) whenever \(\mathcal{B}\) is a lattice basis for \(\mathcal{L}\) if the matrix of \(T\) relative to some, then any, lattice basis for \(\mathcal{L}\), has integer entries and \(|\det T| = 1\). In particular, relative to some fixed lattice basis for \(\mathcal{L}\), the collection of all lattice bases for \(\mathcal{L}\) is in 1-1 correspondence with the group of all such integer matrices.

The routine theory outlined above of lattices in a finite-dimensional real vector space \(V\) takes on a very different flavor when we restrict attention to the real vector space \(\mathbb{C} = \mathbb{R}^2\), denoting a typical lattice in \(\mathbb{C}\) by \(\Lambda\) rather than \(\mathcal{L}\), and we wish to study the collection of \(\Lambda\)-periodic meromorphic functions on \(\mathbb{C}\). We can then no longer use the notion of lattice equivalence mentioned above; instead, we say lattices \(\Lambda_1\) and \(\Lambda_2\) in \(\mathbb{C}\) are complex equivalent if there is a non-zero \(c \in \mathbb{C}\) for which \(\Lambda_2 = c \Lambda_1\) (note that multiplication by a non-zero \(c\) describes the typical invertible linear transformation on \(\mathbb{R}^2\) which is conformal). Then \(f\) is meromorphic and \(\Lambda_2\)-periodic \(\Leftrightarrow g(z) = f(cz)\) is meromorphic and \(\Lambda_1\)-invariant.

Definitions and Notations.
(i) An elliptic function is a non-constant meromorphic function $f$ on $\mathbb{C}$ which is periodic with respect to some lattice $\Lambda$ in $\mathbb{C}$. For $\mathbb{M}$ (capital mu) a lattice in $\mathbb{C}$, $\text{Ell}(\mathbb{M})$ denotes the space of $\mathbb{M}$-periodic elliptic functions.

(ii) There are two widely used notational conventions for elliptic functions.

(1) The first convention replaces $\mathbb{M}$ by $2\Lambda$ and stems from the natural notations for Jacobi's elliptic functions $sn$. Then $2\Lambda$ is a subgroup of $\Lambda$ and the quotient group $\Lambda/2\Lambda$ is isomorphic to the 4-element group $\mathbb{Z}_2 \times \mathbb{Z}_2$. For any lattice basis $\mathcal{B} = \{\lambda_1, \lambda_2\}$ for $\Lambda$ (in the literature, many authors use $\omega_j$ in place of $\lambda_j$), $\{\lambda_0 = 0, \lambda_1, \lambda_2, \lambda_3 = \lambda_1 + \lambda_2\}$ is a complete set of coset representatives for $\Lambda/2\Lambda$. Note that $\{\pm \lambda_1, \pm \lambda_2\} \text{and} \{\pm \lambda_2, \pm \lambda_3\}$ are also lattice bases for $\Lambda$ and, in an obvious sense, $\pm \lambda_i \pm \lambda_j = \pm \lambda_k \mod 2\Lambda$ for $\{i, j, k\} = \{1, 2, 3\}$ expresses the addition operation for non-zero members of $\Lambda/2\Lambda$. It's customary to refer to $\{\lambda_j : j = 1, 2, 3\}$ as the fundamental half-periods for any $f \in \text{Ell}(2\Lambda)$ and to say that $\{E_j = f(\lambda_j) : j = 1, 2, 3\}$ is the set of half-period values for $f$.

(2) The second convention restricts attention to lattices of the form $\Lambda_\tau = \text{span}_\mathbb{Z}\{1, \tau\}$ where $\text{Im} \tau \neq 0$ and we write $\text{Ell}(\tau)$ in place of $\text{Ell}(\Lambda_\tau)$. It's easy to check that every lattice in $\mathbb{C}$ is complex equivalent to a unique lattice of the form $\Lambda_\tau$ with $\text{Im} \tau > 0$ and $\text{Re} \tau \in (0, 1/2)$. However, many authors prefer to sacrifice uniqueness either by allowing $\tau$ to range over the upper half plane or to range over an open half-strip such as $\{\tau \in \mathbb{C} : \text{Im} \tau > 0 \text{ and } \text{Re} \tau \in (-1, 1)\}$. Then $\{1/2, \tau/2, (1+\tau)/2\}$ are the fundamental half-periods for members of $\Lambda_\tau$. Note that Jacobi's $sn$ functions and his related elliptic functions $cn$ and $dn$ belong to $\text{Ell}(\tau)$ with $\text{Re}(\tau)=0$ (where cells are rectangles rather than non-rectangular parallelograms).

(iii) In keeping with (ii), there are two notations for Weierstrass's $p$ functions (the German Fraktur letter $p$ is pronounced "pay" and is often denoted by $\mathcal{P}$ in English texts and handwritten English since those who didn’t grow up using Fraktur letters either find it too difficult to write or too easy to confuse with the ordinary letter $p$). They are defined as follows:

$$p_M(z) = \frac{1}{z^2} + \sum_{\mu \in \mathbb{M}\setminus\{0\}} \left\{ \frac{1}{(z-\mu)^2} - \frac{1}{\mu^2} \right\} \text{ for } M \text{ any lattice in } \mathbb{C};$$

$$p(z, \tau) = p_{\mathbb{M}_\tau}(z) = \frac{1}{z^2} + \sum_{(j, k) \in \mathbb{Z}^2 \setminus \{0\}} \left\{ \frac{1}{(z-(j+\tau k))^2} - \frac{1}{(j+\tau k)^2} \right\} \text{ for } \text{Im}(\tau)>0.$$ 

Obviously, after checking that the series for $p_\mathbb{M}$ converges uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{M}$ or $\mathbb{C}\setminus\Lambda_\tau$, $p_\mathbb{M}$ is a special case of a Mittag-Leffler expansion for a meromorphic function with poles of order 2 at the points in $\mathbb{M}$. But note that $p(z, \tau)$ is bi-holomorphic (separately holomorphic in each variable) on the open domain in $\mathbb{C} \times \mathbb{H}$ where $z \notin \Lambda_\tau$.

The homework exercises outline a proof both of the convergence statement and the non-obvious fact that $p_\mathbb{M} \in \text{Ell}(\mathbb{M})$.

Elementary Properties of $\text{Ell}(2\Lambda)$ for any lattice $\Lambda$ in $\mathbb{C}$.
(i) No member of $\text{Ell}(2\Lambda)$ is holomorphic. Indeed, if $f \in \text{Hol}(\mathbb{C})$ is $2\Lambda$-periodic, then $f$ is bounded on each $2\Lambda$-cell, hence is bounded on $\mathbb{C}$, hence is a constant function by Liouville's Theorem.

(ii) $\mathbb{F}_{2\Lambda} = \text{Ell}(2\Lambda) \cup \{\text{constant functions}\}$ is a subfield of the field $\text{Mero}(\mathbb{C})$.

As in the homework exercises, we may identify $\mathbb{F}_{2\Lambda}$ either with the field of meromorphic functions from the compact Riemann surface $\mathbb{T}_{2\Lambda}$ into $\overline{\mathbb{C}}$ or with the collection $\text{Hol}(\mathbb{T}_{2\Lambda}, \mathbb{S}^2)$ of all holomorphic functions from $\mathbb{T}_{2\Lambda}$ into the compact Riemann surface $\mathbb{S}^2$ (the Riemann sphere in $\mathbb{R}^3$). Recall that while any particular stereographic projection may be used to transport addition and multiplication operations on $\overline{\mathbb{C}}$ to operations on $\mathbb{S}^2$, there's no "natural" way to add and multiply on $\mathbb{S}^2$.

(iii) For each $f \in \mathbb{F}_{2\Lambda}$, $f_{\text{even}}(z) = \frac{1}{2}(f(z) + f(-z))$ defines an even function in $\mathbb{F}_{2\Lambda}$ and $f_{\text{odd}}(z) = \frac{1}{2}(f(z) - f(-z))$ defines an odd function in $\mathbb{F}_{2\Lambda}$ for which $f = f_{\text{even}} + f_{\text{odd}}$.

(iv) For each $f \in \text{Ell}(2\Lambda)$, there is an integer $N \geq 2$ called the order of $f$ such that for each $2\Lambda$-cell $P$ and each $w_0 \in P$, $P$ contains precisely $N$ poles, counting multiplicities, for $f$ and also contains precisely $N$ zeros, counting multiplicities, for $f - w_0$. In this sense, $f$ is "essentially" $N$ to 1 either from $P$ or $\mathbb{T}_{2\Lambda}$ onto either $\overline{\mathbb{C}}$ or $\mathbb{S}^2$.

Proof: First take $P$ to a $2\Lambda$-cell for which $\partial P$ contains no poles for $f$. Using $2\Lambda$-periodicity of $f$, the contributions to $\int_{\partial P} f dz$ from opposite sides of $\partial P$ cancel out, yielding $0 = \int_{\partial P} f dz = \sum (\text{residues of } f \text{ at poles in } \text{int}(P))$. Since we know the total number $N$ of poles of $f$ in $\text{int}(P)$ is $>0$ by (i), we deduce that $N \geq 2$ with $N=2$ $
Rightarrow f$ either has a single pole in $\text{int}(P)$ whose order is 2 and whose residue is 0 or $f$ has exactly 2 poles in $\text{int}(P)$ both of which are simple and their residues have opposite signs.

Now fix $w_0$ in $\mathbb{C}$ and choose a $2\Lambda$-cell $P$ for which $f$ has no poles on $\partial P$ and $f - w_0$ has no zeros on $\partial P$. Then, from above, the line integral along $\partial P$ of the $\text{Ell}(2\Lambda)$ function $\frac{f}{f - w_0}$ is zero. Using the argument principle,

$$0 = \frac{1}{2\pi i} \int_{\partial P} \frac{f'}{f - w_0} dz = (\text{total number of zeros of } f - w_0 \text{ in } \text{int}(P) - (\text{total number of poles of } f \text{ in } \text{int}(P))) \text{ and this proves (iv).}$$

(v) For $f$ and $g$ in $\text{Ell}(2\Lambda)$, $f/g$ is a constant function $\Leftrightarrow f$ and $g$ have the same order and $f/g$ has no zeros (equivalently, $g/f$ has no poles) $\Leftrightarrow f$ and $g$ have precisely the same list, counting multiplicities, of zeros $\Leftrightarrow f$ and $g$ have precisely the same list, counting multiplicities, of poles. Clearly, this statement follows immediately from (i) and (iv).

Fundamental Theorem of Elliptic Functions
(i) For each lattice \( \Lambda \) in \( \mathbb{C} \), \( p = p_\Lambda \) and \( p' \) generate \( \text{Ell}(2\Lambda) \) in the sense that, for each \( f \in \text{Ell}(2\Lambda) \), there are unique rational functions \( R_0 \) and \( R_1 \) such that \( f = R_0 p + p'(R_1 p) \). In particular, the even (respectively, odd) members of \( \text{Ell}(2\Lambda) \) are those for which \( R_1 = 0 \) (respectively, \( R_0 = 0 \)).

(ii) \( f \) is an elliptic function \( \iff \) there is some non-zero \( c \) and some \( \tau \) for which \( f(cz) \) is a rational combination of \( p(z, \tau) \) and \( \frac{dp}{dz}(z, \tau) \).

The proof of (i) is one of the homework exercises and (ii) is immediate from (i) and the above remarks.

Concluding Remarks

(i) As a general principle, when \( S \) is a Riemann surface, the collection of all holomorphic functions from \( S \) into \( S^2 \) (or equivalently, meromorphic functions from \( S \) into \( \mathbb{C}^* \)) is very large and "unmanageable" when \( S \) is non-compact but not very large and "highly structured" when \( S \) is compact. Thus, when \( S = S^2 \), all we get are the functions corresponding to rational functions. The Fundamental Theorem of Elliptic Functions parallels this by showing that, for \( S = \mathbb{C}/2\Lambda \), all we get are rational combinations of \( p \) and \( p' \).

(ii) In group theoretic terms, the Riemann surfaces of the form \( \mathbb{T}/2\Lambda \) are the orbit spaces for discrete, co-compact subgroups of the translation subgroup of \( \text{SL}(2, \mathbb{C})/(\pm I_2) = \text{Möb}(\mathbb{C}) \). Other very nice compact Riemann surfaces arise as the orbit space of other discrete, co-compact, not necessarily abelian, subgroups \( \Gamma \) of \( \text{SL}(2, \mathbb{C}) \). An especially important space for number theoretic applications uses the group \( \Gamma = \text{SL}(2, \mathbb{Z}) \). Books on Riemann surfaces usually go into great detail on analogs of elliptic functions living on certain families of compact Riemann surfaces.

(iii) We don't get anywhere trying to study the collection of all holomorphic or meromorphic functions which are \( \mathbb{Z} \)-periodic (i.e., \( f(z + k) = f(z) \forall (z, k) \in \mathbb{C} \times \mathbb{Z} \)). There are just too many of them, not just rational combinations of the basic \( \mathbb{Z} \)-periodic function \( e^{2\pi i z} \) (rational trig functions) but horrible things such as \( e^{i \theta z} \). In keeping with the above general principle, the problem is that \( \mathbb{C}/\mathbb{Z} \) isn't compact. But special families of \( \mathbb{Z} \)-periodic functions other than the rational trig functions are of great interest in a variety of applications. One such family arises by replacing the Weierstrass \( p(z, \tau) \) function by the Jacobi \( \Theta \) function defined on \( \mathbb{C} \times \mathbb{H} \) by

\[
\Theta(z, \tau) = \sum_{k \in \mathbb{Z}} e^{\pi i k^2 \tau} e^{2\pi i k z}.
\]

Somewhat amazingly, the properties of \( \Theta \) lead to a proof of the number-theoretic fact that any integer \( \geq 2 \) is either the sum of two squares of positive integers or the sum of four squares of positive integers.

(iv) Eisenstein series are infinite series of the form

\[
E(\alpha, \tau) = \sum_{(j,k) \in \mathbb{Z}^2 \setminus \{0,0\}} \frac{1}{(j\tau + k)^\alpha} \text{ for } \text{Re}(\alpha) > 2 \text{ and } \text{Im}(\tau) > 0. \]

These "crop up" with \( \alpha = 3 \) for various formulas involving \( \text{Ell}(\tau) \) but also enter into other special functions and their applications. Some difficult problems remain open precisely because of the difficulty in getting explicit numerical values for certain Eisenstein series.
(v) In brief, the beautiful theory of elliptic functions is the launch pad for investigation of many types of special functions and, more generally, the currently very popular field of analytic number theory.