NOTES ON UNIVALENT FUNCTION THEORY

In complex function theory, the term univalent function means a 1-1 holomorphic function \( f \) on some domain \( D \subset \mathbb{C} \), another description being a conformal equivalence from \( D \) onto an image domain. When \( D \) is a simply connected domain with \( D \neq \mathbb{C} \), the Riemann Mapping Theorem gives a 1-1 correspondence between the univalent functions on \( D \) and the univalent functions on the unit disk \( \mathbb{D} \). In the early part of the 20th century, Kőbe (usually spelled Koebe in English language textbooks) and Bieberbach denoted by \( S \) the class of all univalent functions \( f \) on \( \mathbb{D} \) satisfying the normalization conditions \( f(0) = 0 \) and \( f'(0) = 1 \). Obviously, when \( g \) is univalent on \( \mathbb{D} \), \( f = \frac{g - g'(0)}{g'(0)} \) is in \( S \) and passage from \( g \) to \( f \) eliminates undesirable scaling artifacts of \( g \).

Kőbe, Bieberbach, and other German mathematicians referred to members of \( S \) as being schlicht. English translations of the German word schlicht include "simple", "straightforward", and "standardized". But most authors don't bother to pick one of these translations and instead continue to describe \( S \) as the class of schlicht functions.

Closely related to \( S \) is the class \( \Sigma \) of all univalent functions \( h \) on \( \mathbb{D} \setminus \{0\} \) having a simple pole at \( 0 \) with \( \text{Res}(h,0) = 1 \). It follows that \( f \in S \iff h = 1/f \) is in \( \Sigma \).

Also, when \( f \in S \) and \( h \in \Sigma \), we have coefficient sequences \( (a_k)_{k \geq 2} \) and \( (c_k)_{k \geq 0} \) for which

\[
\begin{align*}
  f(z) &= z + \sum_{k=2}^{\infty} a_k z^k \quad (\text{convergence } \forall z \in \mathbb{D}) \quad (1) \\
  h(z) &= 1/z + \sum_{k=0}^{\infty} c_k z \quad (\text{convergence } \forall z \in \mathbb{D} \setminus \{0\}). \quad (2)
\end{align*}
\]

In manipulating such series, it's convenient to use the shorthand h.o.t. for higher order terms in \( z \). Thus, \( f(z) = z(1 + a_2 z + \text{h.o.t.}) \) and, when \( h = 1/f \), we have

\[
h(z) = (1/z)(1 - a_2 z + \text{h.o.t.}) = 1/z + c_0 + \text{h.o.t.}
\]

with \( c_0 = -a_2 \). A very useful trick in univalent function theory is to pass from any \( f \in S \) to the unique \( g \in S \) for which

\[
g \text{ is an odd function and } g(z)^2 = f(z^2). \quad (3)
\]

This is done as follows. We can inductively construct the unique sequence \( (b_k)_{k \geq 2} \) of coefficients for which, with \( \psi(z) = 1 + \sum_{k=1}^{\infty} b_k z^k \), we have \( (\psi(z))^2 = f(z)/z \). In shorthand notation,

\[
(1 + b_1 z + b_2 z^2 + \text{h.o.t.})^2 = 1 + 2b_1 z + (2b_2 + b_1^2) z^2 + \text{h.o.t.} = 1 + a_2 z + a_3 z^2 + \text{h.o.t.}
\]

so \( b_1 = a_2/2 \) and \( b_2 = (a_3 - b_1^2)/2 \), etc. We then define \( g(z) = z\psi(z^2) \) and check easily that \( (g(z))^2 = f(z^2) \).

Very special members of \( S \) are the Kőbe function \( k(z) = \frac{z}{(1+z)^2} \) and the rotated Kőbe functions \( k_\theta(z) = e^{-i\theta} k(e^{i\theta} z) = \frac{e^{i\theta} z}{(1+e^{i\theta} z)^2} \), \( \theta \in [0, 2\pi] \). Note that when \( k_\theta \) is expanded as in (1), we have \( |a_2| = |-2e^{i\theta}| = 2 \). In 1908, Kőbe proved that there is a
fixed constant $K>0$ for which $D(0,K) \subset f(D) \forall f \in \mathcal{S}$ and speculated that the largest choice for $K$ is $1/4$ with $D(0,1/4)$ not contained in $f(D) \iff f = k\theta$ for some $\theta$. In 1916, Bieberbach proved that these speculations were correct; this result is known as the Kôbe-Bieberbach Viertel Satz (1/4 Theorem).

In brief, Bieberbach’s method was as follows. He first proved that, for $h \in \Sigma$ with the Laurent expansion (2) and with $A(r)$ the area of the interior of the simple closed curve $\Gamma(r) = h(C(0,r))$ for $0 < r < 1$, then

$$\sum_{k=1}^{\infty} k|c_k|^2 \leq 1 \text{ with equality } \iff \lim_{r \to 1} A(r) = 0 \quad (4)$$

This result is known as Bieberbach’s area theorem. Passing from $f \in \mathcal{S}$ with the power series expansion (1) to the odd functions $g \in \mathcal{S}$ satisfying (3) and then applying the area theorem to $h = 1/g$, algebraic computations show that $|a_2| \leq 2$ with equality $\iff f = k\theta$ for some $\theta \in [0,2\pi)$. Finally, an easy trick using this $a_2$ theorem proves the Viertel Satz by showing that if $f \in \mathcal{S}$ and $w \not\in f(D)$, we can construct a function in $\mathcal{S}$ with an expansion $z + (a_2 + 1/w)z^2 + h.o.t.$ and deduce from both $|a_2|$ and $|a_2 + 1/w|$ being $\leq 2$ that $|w| \geq 4$ with equality $\iff |a_2| = 2$.

Based on his success with the $a_2$ theorem, Bieberbach was led to conjecture that, for $f \in \mathcal{S}$ described by (1), $|a_n| \leq n \forall n \geq 3$. This quickly became known as the Bieberbach Conjecture and occupied much of the attention of the world’s best complex function theorist for 70 years until it was finally proved to be correct in the 1980s. There remain open several other coefficient problems (including the size of the $c_k$ coefficients in (2) for functions in the class $\Sigma$) as well as a number of other research questions about univalent functions.