Rigidity of Geodesic Flows on Negatively Curved Manifolds of Dimensions 3 and 4

by Renato Feres and Anatole Katok[1]

Mathematics 253-37
California Institute of Technology
Pasadena, CA 91125 USA

1. Formulation of Results.

Throughout this paper M will denote a C^∞ compact manifold without boundary with a Riemannian metric σ of negative sectional curvature.

Recently Masahiko Kanai [4] proved that if the stable horospheric foliation W^s on the unit tangent bundle $V = SM$ is C^∞ and the curvature K satisfies the following pinching condition

$$-9/4 < K \leq -1 \tag{1}$$

then the geodesic flow φ_t for the metric σ is C^∞ isomorphic to the geodesic flow for a metric of constant negative curvature.

[1] Partially supported by NSF Grant DMS 85-14630

In the present note we improve Kanai's result by removing the pinching condition in dimension three and making it sharp in dimension four. More precisely, we prove the following theorems.

<u>Theorem 1</u>. If dim M = 3 and the stable horospheric foliation W^s on the unit tangent bundle of M is C^∞ then the geodesic flow for the metric σ is C^∞ isomorphic to the geodesic flow for a metric of constant negative curvature.

<u>Theorem 2</u>. If M is four-dimensional, the stable horospheric foliation is C^∞ and the sectional curvature is restricted to the interval $-4 < K \leq -1$ then its geodesic flow is C^∞ isomorphic to the geodesic flow for a metric of constant negative curvature.

Theorem 1 was proved by the second author and Theorem 2 by the first one.

<u>Corollary</u>. Under the assumptions of either Theorem 1 or Theorem 2 the topological entropy of the geodesic flow for the metric σ is equal to the metric entropy with respect to Liouville measure.

We observe that the pinching assumption on K in Theorem 2 is optimal since the geodesic flow on the complex hyperbolic plane has smooth horospheric foliations and $K_{minimum}/K_{maximum} = 4$ holds.

The results of this paper were obtained when both authors were visiting the Sonderforschungsbereich "Geometrie und Analysis" at the Mathematics Institute of the University of Göttingen, West Germany. We feel a pleasant obligation to thank the Sonderforschungsbereich for its financial support, the Institute für Mathematische Stochastik of the University of Göttingen for providing working facilities and secretarial help, and

especially the organizers of the SFB program in geodesic flows, Manfred Denker and Paddy Patterson for creating a very stimulating working environment.

2. Introduction.

In what follows we summarize the basic constructions used in the paper. The reader may wish to consult [4] for further details.

Let $TV = E^{o} + E^{-} + E^{+}$ be the hyperbolic (Anosov) splitting for the geodesic flow φ_t where E^{o} is the one-dimensional subbundle generated by the flow direction and E^{-}, E^{+} are correspondingly contracting and expanding sub-bundles, i.e. the distributions tangent to the stable horospheric foliation W^s and the unstable horospheric foliation W^u. In general, the bundles E^{+} and E^{-} are not very smooth. It is believed that they may even not be C^1, although no examples are known; a pinching condition for the curvature $-k^2 < K \leq -1$ guarantees that E^{+} and E^{-} are of class $C^{\frac{2}{k}}$. Both bundles have the same smoothness because the differential of the "flip" map $J: V \to V$, $J(v) = -v$ interchanges them. Our main assumption that forces rigidity is that one of the bundles (and hence both of them) is C^{∞}.

Let $\tilde{M}$ be the universal cover of M provided with the lift $\tilde{\sigma}$ of the Riemannian metric σ. The fundamental group Γ of M acts on $\tilde{M}$ as a group of isometries. Let B be the sphere at infinity (ideal boundary) for $\tilde{M}$. Every two distinct points b^{-}, $b^{+} \in B$ can be connected by a unique geodesic on $\tilde{M}$ so that the space P of geodesics on $\tilde{M}$ is naturally identified with the space $B \times B$ minus its diagonal. The group Γ acts on P in a natural way. This action has dense orbits and the fixed points of its elements, which correspond to closed geodesics on M, are dense in B. The "flip" map is also projected naturally to the space P. We will use the same notation J for it. Thus if $(b_1, b_2) \in P$, $J(b_1, b_2) = (b_2, b_1)$.

On the unit tangent bundle V there is a natural φ_t-invariant 1-form α (the contact form) whose kernel coincides with $E^{+} + E^{-}$. The restriction of $d\alpha$ to that kernel is a non-degenetate φ_t-invariant 2-form. Its lift to $\tilde{M}$ produces a Γ-invariant 2-form on the unit tangent bundle $\widetilde{V}$ of $\widetilde{M}$ which, due to its invariance with respect to the geodesic flow, can be projected to a Γ-invariant symplectic form Ω on P. Let us denote the projections of the stable and unstable bundles to P by F^{+} and F^{-}, and the projection of the horospheric foliations W^s and W^u there by $\mathcal{F}^s$ and $\mathcal{F}^u$. The last two foliations are transversal C^{∞} Lagrangian foliations of the symplectic manifold (P, Ω). Kanai calls the quadruple $(P, \Omega, \mathcal{F}^s, \mathcal{F}^u)$ a <u>bipolarized</u> <u>symplectic</u> <u>manifold</u>.

Following [4] we can define in $(P, \Omega, \mathcal{F}^s, \mathcal{F}^u)$ an affine connection ∇ characterized by the following properties: (i) ∇ is a torsion-free affine connection, (ii) Ω is parallel, that is $\nabla\Omega = 0$, and (iii) if f is a smooth function defined locally on P, which is constant on each leaf of $\mathcal{F}^s$ (resp. $\mathcal{F}^u$), then $\nabla_\xi df = 0$ for any $\xi \in F^{-}$ (resp. F^{+}). Denote by R the curvature tensor of ∇. Define

$$\check{R}(\xi_1, \xi_2, \xi_3, \xi_4) = \Omega(R(\xi_1, \xi_2)\xi_3, \xi_4)$$
$$\omega(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = (\nabla_{\xi_1} \check{R})(\xi_2, \xi_3, \xi_4, \xi_5), \quad \xi_i \in TP. \tag{2}$$

We have $\nabla R = 0$ iff $\omega = 0$. Also $\mathcal{F}^s$ (resp. $\mathcal{F}^u$) is flat, that is $R(\xi_1, \xi_2) = 0$ whenever ξ_1 and ξ_2 belong to F^{-} (resp. F^{+}).

Note that the hypothesis on smoothness of $F^{\pm}$ is required for the above tensor fields to be well defined. It actually requires only a certain finite degree of smoothness but we will not be concerned with this matter here.

Kanai uses condition (1) and a rather simple dynamical trick to prove that the tensor ω, hence ∇R, vanishes ([4] Proposition 2.4).

In Section 3 of his paper Kanai classifies symmetric bipolarized symplectic manifolds.

Based on that classification he deduces in Section 4 the rigidity of the geodesic flow. Thus, in order to prove Theorems 1 and 2 it is enough to establish that under the weaker pinching assumptions of these theorems the form ω still vanishes and then refer to ([4], Sections 3 and 4). We would like to point out that we have to use various dynamical properties of the geodesic flow in a more elaborate way than does Kanai in the proof of his Proposition 2.4.

Proposition. Under the hypothesis of either Theorem 1 or Theorem 2 we have $\omega \equiv 0$.

We state below a number of formal properties of the tensor ω which will be used later on. For simplicity we write $(1 \cdots 5)$ instead of $\omega(\xi_1, \cdots, \xi_5)$.

2.1) $(12345) = -(13245)$

2.2) $(12345) = (12354)$

2.3) $(12345) + (13425) + (14235) = 0$

2.4) $(12345) - (14523) = (15342) - (14253)$

2.5) $(12345) = (21345) + (32145)$

2.6) $\omega(\xi_1, \ldots, \xi_5)$ vanishes whenever at least one of the pairs (ξ_2, ξ_3) and (ξ_4, ξ_5) belongs to a same subbundle F^- or F^+.

2.7) ω is invariant under Γ.

2.8) J is an affine map with respect to ∇; $J^*\Omega = -\Omega$, $J^*\omega = -\omega$.

We will sketch the proofs for some of the properties in order to give the flavor of the arguments involved. A basic fact to have in mind is that ∇ preserves the distributions F^+ and F^-, that is, if ξ is a vector field tangent to F^+ (resp. F^-) and ξ' an arbitrary vector field, then $\nabla_{\xi'}\xi$ is a field tangent to F^+ (resp. F^-).

Consider, for example, property (2.2). The connection ∇ can be extended in the usual fashion to a derivation of the mixed tensor algebra of TP. It follows that, for vector fields ξ and η, $R(\xi, \eta)$ is also a derivation with the further property that it annihilates scalar functions. Since Ω is parallel, we have for fields ξ_1, ξ_2 that

$$0 = R(\xi, \eta)\, \Omega(\xi_1, \xi_2) = \Omega(R(\xi, \eta)\xi_1, \xi_2) + \Omega(\xi_1, R(\xi, \eta)\xi_2).$$

The antisymmetry of Ω yields $\Omega(R(\xi, \eta)\xi_1, \xi_2) = \Omega(R(\xi, \eta)\xi_2, \xi_1)$. This is a property of $\check{R}$ similar to (2.2). If we now use the fact that ∇ preserves the foliations and that, for a (0, k)-tensor T and vector fields ξ_i, $(\nabla_\xi T)(\xi_1, \cdots, \xi_k) = \xi T(\xi_1, \cdots, \xi_k) - \sum_i T(\xi_1, \cdots, \nabla_\xi \xi_i, \cdots, \xi_k)$, we obtain (2.2) from the previously derived property of $\check{R}$.

Properties (2.1), (2.3), and (2.4) are derived in a similar way. Notice that (2.4) is the symplectic counterpart of the symmetry (1234) = (3412) of the curvature tensor associated to a Riemannian connection. The proof of property (2.5) is certainly more demanding but also straightforward. Property (2.6) follows from the fact that the leaves of $\mathcal{F}^u$ and $\mathcal{F}^s$ are flat. The other properties are also easily derived. $\square$

Using the flip mapping J and properties 2.1 and 2.2 we see that, in order to prove the Proposition it suffices to establish that $\omega(\xi_1^+, \xi_2^-, \xi_3^+, \xi_4^-, \xi_5^+) = 0$, where ξ_i^+ (resp. ξ_i^-) denotes an arbitrary vector in F^+ (resp. F^-). Notice that the properties 2.1–2.6 allow us to permute any two entries of $\omega(\xi_1^+, \xi_2^-, \xi_3^+, \xi_4^-, \xi_5^+)$ of same sign.

3. Resonances at periodic points.

Assume $\omega \neq 0$ and consider the set A' of points of P where ω does not vanish. A' is an open, Γ-invariant, J-invariant set. Since the action of Γ on P is topologically transitive, A'

must be dense in P. Consider the set A of points of A' which are images of periodic points for the geodesic flow on SM. For each such point there exists an element $\gamma \in \Gamma$ that fixes it. Note that this set of points is J-invariant and dense.

For $\gamma \in \Gamma$ let us denote the action of γ on P by A_γ. For each point $p \in A$ there exists an element $\gamma \in \Gamma$ such that $A_\gamma p = p$. Let us fix $p \in A$. In what follows we suppress the dependence on p in our notation. Let us denote the differential of the map A_γ at p by $\rho : T_pP \to T_pP$. This map is symplectic, i.e. it preserves the 2-form Ω. Since its eigenvalues are different from one in absolute value they split into pairs of mutually inverse numbers. We denote those eigenvalues $\lambda_1, \ldots, \lambda_{n-1}, \lambda_1^{-1}, \ldots, \lambda_{n-1}^{-1}$ so that

$$1 < |\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_{n-1}|$$

If λ_i is real we will denote its root space by F_i^+ and the root space of λ_i^{-1} by F_i^-. If λ_i is complex we will denote by F_i^+ (corr. F_i^-) the real part of the sum of the root spaces corresponding to λ_i and $\bar{\lambda}_i$ (corr. to λ_i^{-1} and $\bar{\lambda}_i^{-1}$). In all cases we will call the elements of the spaces $F_i^\pm$ root vectors associated to $\lambda_i^{\pm 1}$.

Notice that 1/4 pinching of the sectional curvature of M implies that $|\lambda_i| < |\lambda_j \lambda_k|$ $\forall_{i,j,k}$.

<u>Lemma 1</u>. Suppose $p \in A$ and let $\xi_1, \xi_3, \xi_5 \in F^+$, $\xi_2, \xi_4 \in F^-$ be such that $\omega(\xi_1, \ldots, \xi_5) \neq 0$. For $n = 4$, assume further that $|\lambda_i| < |\lambda_j \lambda_k|$ for any i, j, k. Then at least one of the following relations must take place:

$n = 3$	ξ_1	ξ_2	ξ_3	ξ_4	ξ_5 belong to	
1) $\lvert\lambda_1^3\rvert = \lvert\lambda_2^2\rvert,\ \lvert\lambda_1\rvert < \lvert\lambda_2\rvert$	F_1^+	F_2^-	F_1^+	F_2^-	F_1^+	
2) $\lvert\lambda_1^2\rvert = \lvert\lambda_2\rvert,\ \lvert\lambda_1\rvert < \lvert\lambda_2\rvert$	F_1^+	F_1^-	F_1^+	F_2^-	F_1^+	or
	F_1^+	F_2^-	F_2^+	F_2^-	F_1^+	
$n = 4$						
1) $\lvert\lambda_1^2\lambda_2\lambda_3^{-2}\rvert = 1,\ \lvert\lambda_1\rvert < \lvert\lambda_2\rvert < \lambda_3\rvert$	F_1^+	F_3^-	F_1^+	F_3^-	F_2^+	
2) $\lvert\lambda_1\lambda_2^2\lambda_3^{-2}\rvert = 1,\ \lvert\lambda_1\rvert < \lvert\lambda_2\rvert < \lvert\lambda_3\rvert$	F_1^+	F_3^-	F_2^+	F_3^-	F_2^+	
3) $\lvert\lambda_1^3\lambda_2^{-1}\lambda_3^{-1}\rvert = 1,\ \lvert\lambda_1\rvert < \lvert\lambda_2\rvert < \lvert\lambda_3\rvert$	F_1^+	F_2^-	F_1^+	F_3^-	F_1^+	
4) $\lvert\lambda_1^3\lambda_2^{-2}\rvert = 1,\ \lvert\lambda_1\rvert < \lvert\lambda_2\rvert$	F_1^+	F_2^-	F_1^+	F_2^-	F_1^+	
5) $\lvert\lambda_1^3\lambda_3^{-2}\rvert = 1,\ \lvert\lambda_1\rvert < \lvert\lambda_3\rvert$	F_1^+	F_3^-	F_1^+	F_3^-	F_1^+	
6) $\lvert\lambda_2^3\lambda_3^{-2}\rvert = 1,\ \lvert\lambda_2\rvert < \lvert\lambda_3\rvert$	F_2^+	F_3^-	F_2^+	F_3^-	F_2^+	

Proof. Recall that $A_\gamma^* \omega = \omega$. If ξ_i, $i = 1, \ldots, 5$ are root vectors of ρ associated to eigenvalues μ_i, then

$$\omega_p(\xi_1, \ldots, \xi_5) = \lim_{n\to\infty} (A_\gamma^{n*}\omega)_p\, (\xi_1, \ldots, \xi_5) = \lim_{n\to\infty} (\rho^n \xi_1, \ldots, \rho^n \xi_5)$$

$$= \lim_{n\to\infty} \left(\textstyle\prod_{i=1}^5 \mu_i\right)^n \omega_p(\xi_1, \ldots, \xi_5).$$

Hence $\lvert\prod_{i=1}^5 \mu_i\rvert = 1$, unless $\omega_p(\xi_1, \ldots, \xi_5) = 0$. In particular if $\xi_1, \ldots, \xi_5$ are as stated in the lemma, one has $\lvert\lambda_{\ell_1}\lambda_{\ell_3}\lambda_{\ell_5}\rvert = \lvert\lambda_{\ell_2}\lambda_{\ell_4}\rvert$. We observe in passing that for $n = 3$, λ_1 and λ_2 are distinct and real. The relations given above then follow from the last equation by explicitly enumerating the possibilities when n is equal to 3 or 4. For the right hand side of the table, which is presented there for the sake of convenience, one should bear in mind that we can permute the arguments of ω that are labeled with a same sign. $\square$

Remark. Denote by P_i the set of points of A for which relation (i) holds (if $n = 3$, $i \in \{1, 2\}$; for $n = 4$, $i \in \{1, \ldots, 6\}$). For $n = 3$ we clearly have $P_1 \cap P_2 = \emptyset$, otherwise $\lambda_1(p)$ and $\lambda_2(p)$ would be equal to 1 for some p, which is impossible. When $n = 4$, various relations may occur simultaneously. For example, a point p may belong to $P_4 \cap P_1$ if $|\lambda_1|^{35} = |\lambda_2|^{28} = |\lambda_3|^{20}$. On the other hand, the simultaneous occurrence of, say relations (2) and (4) would imply $|\lambda_1|^2 = |\lambda_3|$, which violates the pinching assumption $|\lambda_i| < |\lambda_j\lambda_k|$ $\forall i, j, k$. The diagram below shows which relations may occur simultaneously by connecting the corresponding numbers, i.e., (i)–(j) means $P_i \cap P_j \neq \emptyset$.

$$\begin{array}{ccccccc} (4) & — & (1) & — & (3) & — & (6) \\ & & & & | & & \\ & & & & (2) & & \end{array}$$

Notice that $P_5 \cap P_4$ (resp. $P_5 \cap P_6$) may be nonempty only if the eigenvalue λ_3 (resp. λ_1) has multiplicity 2, in which case relations (5) and (4) (resp. (5) and (6)) coincide; $P_5 \cap P_i = \emptyset$ for $i \in \{1, 2, 3\}$.

4. Proof of the Proposition for 3-dimensional manifolds.

Lemma 2. In the 3-dimensional case the splitting of $F_p^{\pm}$, $p \in A$, into the eigenspaces of $\lambda_1^{\pm 1}(p)$ and $\lambda_2^{\pm 1}(p)$ extends to a smooth splitting of $F^{\pm}$ on an open and dense subset of P.

Proof. The closures of the sets P_i, $i = 1, 2$, which we denote by $\overline{P_i}$, are Γ-invariant and $\overline{P_1} \cup \overline{P_2} = P$ since $P_1 \cup P_2 = A$ is a dense subset of P. Since orbits of Γ are dense either P_1 or P_2 is itself a dense subset of P.

1) Assume that P_1 is dense in P and consider a point p of P_1. Choose vectors ξ_1^+, ξ_3^+, $\xi_5^+ \in F_p^+$ and ξ_2^-, $\xi_4^- \in F_p^-$ for which $\omega_p(\xi_1^+, \xi_2^-, \xi_3^+, \xi_4^-, \xi_5^+) \neq 0$. Define smooth

extensions $\tilde{\xi}_i^{\pm}$ of $\xi_i^{\pm}$ to an open neighborhood of p in such a way that $\tilde{\xi}_i^{\pm} \in F^{\pm}$. By considering a smaller neighborhood U_p if necessary, we may assume that $\omega_q(\tilde{\xi}_1^+, \dots \tilde{\xi}_5^+) \neq 0$ for every $q \in U_p$. Define $L(q) = \{\eta \in F_q^+ : \omega_q(\eta, \tilde{\xi}_2^-, \tilde{\xi}_3^+, \tilde{\xi}_4^-, \tilde{\xi}_5^+) = 0\}$. Then $q \to L(q)$, $q \in U_p$, defines a smooth one-dimensional distribution which agrees with F_2^+ on $P_1 \cap U_p$ (see the right-hand side of the table in Lemma 1). Since $p \in P_1$ was arbitrary, we have thus defined an extension of F_2^+ to a smooth field of lines on an open and dense subset U of P.

Consider now $L'(q) = \{\eta \in F_q^- : \Omega(\eta, L(q)) = 0\}$, $q \in U$, the skew-orthogonal complement of L(q) with respect to the symplectic form. Since Ω is non-degenerate, $q \to L'(q)$ defines a smooth field of lines which, by the Γ-invariance of Ω, must coincide with F_1^- at the points of P_1. We have thus extended F_1^- to a smooth distribution on that same open and dense subset of P. By applying the derivative J_* of the flip mapping to the distributions we have just constructed, we obtain extensions of the remaining F_2^- and F_1^+ on the open and dense subset $U \cap JU$.

2) Now assume that P_2 is dense in P. For $q \in P$ define $L(q) = \{\eta \in F_q^- : \omega_q(F^+, \eta, F^+, \eta, F^+) = 0\} = \{\eta \in F_q^- : \omega(\xi_1, \eta, \xi_2, \eta, \xi_3) = 0, \forall \xi_1, \xi_2, \xi_3 \in F_q^+\}$. Notice that L(p) always contains F_1^- for points $p \in P_2$. In order to see what else is contained in L(p) consider $\eta = a\xi_1^- + b\xi_2^- \in L(p)$ where, as before, $\xi_i^{\pm}$ denotes a nonzero vector in $F_i^{\pm}$. We then have $0 = \omega(F^+, \eta, F^+, \eta, F^+) = 2ab\omega(F^+, \xi_1^-, F^+, \xi_2^-, F^+) + b^2\omega(F^+, \xi_2^-, F^+, \xi_2^-, F^+)$. According to the table in Lemma 1 we have either (i) $\omega(F_1^+, F_1^-, F_1^+, F_2^-, F_1^+) \neq 0$ and $\omega(F_1^+, F_2^-, F_1^+, F_2^-, F_1^+) = 0$, or (ii) $\omega(F_1^+, F_2^-, F_2^+, F_2^-, F_1^+) \neq 0$ and $\omega(F_1^+, F_1^-, F_2^+, F_2^-, F_1^+) = 0$. Case (i) implies $ab = 0$ and case (ii), $b = 0$. Therefore L(p) is either $F_1^- \cup F_2^-$ or F_1^-.

Since L(q) is the zero set of a system of quadratic equations which depends smoothly on $q \in P$, there will be an open and dense subset U of P where each component of L(q) varies smoothly in the space of irreducible subvarieties of $L^-(q)$ of corresponding degree, while L(q) coincides with either $F_1^- \cup F_2^-$ or F_1^- at $q \in U \cap P_2$. In any case we are able to extend $F_1^-(p)$,

$p \in P_2$, to a smooth line field on an open and dense subset of P. Using Ω and J as before we obtain extensions for the other directions. This concludes the proof of Lemma 2. $\square$

Denote by $\mathcal{F}_i^s$ (resp. $\mathcal{F}_i^u$), i = 1, 2, the integral foliations associated to the one-dimensional distributions F_i^- (resp. F_i^+) obtained in Lemma 2, which are Γ-invariant and smooth, and are defined on an open dense subset $Q \subset P$. Q can be chosen to be Γ and J-invariant. Recall that $F^{\pm} = F_1^{\pm} + F_2^{\pm}$ and $F^+ = T\mathcal{F}^u$, $F^- = T\mathcal{F}^s$.

<u>Lemma</u> 3. The foliations $\mathcal{F}^s$ and $\mathcal{F}_1^u$ form an integrable pair.

<u>Proof</u>. Denote by $\mathcal{F}_p^s$, $\mathcal{F}_{i,p}^u$, etc., the local leaves of the corresponding foliations that contain the point p. Let $p \in Q$, $q \in \mathcal{F}_p^s$, $p' \in \mathcal{F}_p^u$. Then the local leaves $\mathcal{F}_q^u$ and $\mathcal{F}_{p'}^s$ have a unique intersection point which we will denote by $H_{p,q}p'$. The map $H_{p,q} : \mathcal{F}_p^u \to \mathcal{F}_q^u$ is called a canonical or holonomy map. For the integrability of $\mathcal{F}^s$ and $\mathcal{F}_1^u$ at the point p it is enough to show that

$$DH_{p,q}\, F_{1,p}^u = F_{1,q}^u. \tag{3}$$

Since integrability is a closed condition it is enough to establish it on the dense subset $Q \cap A$ of P. Recall that $A = P_1 \cup P_2$ (see Remark in section 3). For $\gamma \in \Gamma$, recall that A_γ is the action of γ on P. By the definition of the set A, each $p \in Q \cap A$ is the unique fixed point of a map A_γ for some γ.

Since F_2^+ corresponds to the direction of "fast" expansion while F_1^+ is the "slowly" expanding direction, for every vector $\xi \in F_p^+ \backslash F_{1,p}^+$ the direction of the vectors $A_{\gamma *}^n \xi$ exponentially converges to $F_{2,p}^+$ as $n \to \infty$.

On the other hand, canonical maps are γ-invariant which implies that

$$DH_{p,A_\gamma^n q} F_{1,p}^+ = DA_\gamma^n \, DH_{p,q} \, F_{1,p}^+ \tag{4}$$

As $n \to \infty$, $A_\gamma^n q \to p$ so that the left-hand side of (4) converges to $F_{1,p}^+$; hence the same is true for the right-hand side. In particular, the growth rate of $DA_\gamma^n \xi$ for $\xi \in DH_{p,q} F_{1,p}^+$ as $n \to \infty$ is the same as in $F_{1,p}^+$ and hence slower than in $F_{2,x}^+$. But since the spaces $DA_\gamma^n F_q^+$ converge exponentially to F_p^+ as $n \to \infty$ the direction of every vector from $F_q^+ \backslash DH_{p,q} F_{1,p}^+$ will converge to $F_{2,p}^+$. Thus if $F_{1,q}^+ \neq DH_{p,q} F_{1,p}^+$ the distribution F_2^+ which is A_γ-invariant is discontinuous at p, which is not the case. This proves (3). □

Lemma 4. There exists an open dense subset of P such that near each of its points one can define a smooth coordinate system (x_1, x_2, y_1, y_2) whose coordinate lines coincide with the integral curves of the foliation $\mathfrak{F}_1^u$, $\mathfrak{F}_2^u$, $\mathfrak{F}_1^s$, $\mathfrak{F}_2^s$ respectively, and such that $\Omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

Proof. We will prove that such a coordinate system can be defined near each point of $A \cap Q$.

Let γ be any element of Γ. According to Lemma 3 there is a 3-dimensional foliation $\mathcal{G}$ tangent to $F^- + F_1^+$ and one can consider the factor-map induced by A_γ on the space of leaves of that foliation. That factor-map is a one-dimensional hyperbolic expansion, and consequently, it can be linearized by a C^∞ choice of coordinates. Since the distribution F_2^+ is transversal to $\mathcal{G}$ and A_γ-invariant, the linearization yields a vector field ξ_2^+ contained in F_2^+ such that

$$A_{\gamma *} \xi_2^+ = \lambda \xi_2^+ \tag{5}$$

for a constant $\lambda > 1$. Now we can use the invariance of the 2-form Ω to normalize F_2^-, namely we choose ξ_2^- contained in F_2^- such that

$$\Omega(\xi_2^+, \xi_2^-) \equiv 1 \tag{6}$$

Using the A_γ-invariance of Ω and (5) one has

$$A_{\gamma *}\, \xi_2^- = \lambda^{-1}\, \xi_2^-.$$

Now comes the critical point in the argument. According to Lemma 1, if $x \in A$, then at least one of the following conditions takes place:

a) $\omega_x(F_1^+, F_2^-, F_1^+, F_2^-, F_1^+) \not\equiv 0$

b) $\omega_x\,(F_1^+, F_1^-, F_1^+, F_2^-, F_1^+) \not\equiv 0$

c) $\omega_x\,(F_1^+, F_2^-, F_2^+, F_2^-, F_1^+) \not\equiv 0$

Case (a) occurs at points of P_1 and cases (b) and (c) at points of P_2. Denote by P_b and P_c the sets of points of A at which relations (b) and (c), respectively, hold. As in the beginning of the proof of Lemma 2, we conclude that either P_1, P_b, or P_c must be a dense subset of P. We will show that each case leads to a contradiction.

a) Suppose P_1 is dense in P. Define vector fields ξ_1^+ and ξ_1^- on Q, tangent to $\mathcal{F}_1^u$ respectively, so that

$$\omega(\xi_1^+, \xi_2^-, \xi_1^+, \xi_2^-, \xi_1^+) \equiv 1 \tag{7}$$

$$\Omega(\xi_1^-, \xi_1^+) \equiv -1 \tag{8}$$

Using the invariance of the tensors with respect to A_γ one immediately obtains $A_{\gamma *}\, \xi_1^+ = \lambda^{2/3}\, \xi_1^+$, $A_{\gamma *}\, \xi_1^- = \lambda^{-2/3}\xi_1^-$. Thus, all vector fields are transformed by A_γ with constant coefficients. This allows to calculate their brackets, e.g.,

$$A_{\gamma*}[\xi_2^+, \xi_1^-] = [A_{\gamma*}\,\xi_2^+, A_{\gamma*}\,\xi_1^-] = \lambda^{1/3}[\xi_2^+, \xi_1^-] \Rightarrow [\xi_2^+, \xi_1^-] = 0$$

since $\lambda^{1/3}$ is different from the four eigenvalues λ, λ^{-1}, $\lambda^{2/3}$, $\lambda^{-2/3}$. In fact, it can be shown in an analogous way that the other brackets also vanish, so that the vector fields defined above are the coordinate fields for the coordinate system we were looking for. Furthermore, one easily checks that for every pair of our vector fields except for ξ_1^+, ξ_1^- and ξ_2^+, ξ_2^- the form Ω vanishes. This together with (6) and (8) concludes the proof in this first case.

b) and c) The proofs for these two cases go along similar lines as in (a), after doing the obvious changes in relation (7). There are important differences, however. In this case we obtain vector fields ξ_i^+, ξ_i^- for i = 1 and 2, satisfying

$$A_{\gamma*}\,\xi_1^{\pm} = \lambda^{\pm 1/2}\,\xi_1^{\pm},\ A_{\gamma*}\,\xi_2^{\pm} = \lambda^{\pm 1}\,\xi_2^{\pm}.$$

After computing the various bracket relations as was done in the first case, one observes that all brackets vanish except, possibly, for

$$[\xi_2^+, \xi_1^-] = c\xi_1^+ \tag{9}$$

$$[\xi_2^-, \xi_1^+] = c'\xi_1^- \tag{10}$$

An application of the Jacobi identify and relations (9) and (10) yields $cc'\xi_1^- = [\xi_2^-, [\xi_2^+, \xi_1^-]] = -[[\xi_2^-, \xi_2^+], \xi_1^-] - [\xi_2^+, [\xi_1^-, \xi_2^-]] = 0$, so that $cc' = 0$. Observe now that J interchanges the foliations $\mathfrak{F}_i^u$ and $\mathfrak{F}_i^s$ so that $J_*\,\xi_i^+ = f_i\xi_i^-$, $J_*\xi_i^- = g_i\xi_i^+$ for functions f_i, g_i on the J-invariant set Q. Notice that $\xi_i^+ = (J^2)_*\xi_i^+ = f_ig_i\xi_i^+$ so that f_i and g_i never vanish. By applying J_* to the relations (9) and (10) one obtains $c' = (f_2\,g_1/f_1)\,c$, hence $c = 0$ iff $c' = 0$. Therefore $c = c' = 0$ and we arrive at the same conclusion as in case (a). □

<u>End of the proof of Proposition for</u> n = 3. In the coordinate system defined in Lemma 4, we

have

$$\mathcal{F}^s = \{x_1 = \text{const.}, x_2 = \text{const.}\}, \ \mathcal{F}^u = \{y_1 = \text{const.}, y_2 = \text{const.}\},$$

and $\Omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. A simple calculation shows that, in this case ∇ is flat on the open and dense set Q and hence on the whole P. Therefore $\omega \equiv 0$, contradicting the original assumption. □

5. Proof of the proposition for 4-dimensional manifolds.

We continue to use the notations of Section 4 but naturally assume that dim M = 4. Recall that P_i denotes the set of points in A for which relation (i) in the table of Lemma 1 holds, $i \in \{1, 2, 3, 4, 5, 6\}$. Since A is dense in P we must have $P = \bigcup_i \overline{P_i}$. Also each of $\overline{P_i}$ is Γ-invariant, hence at least one of them must coincide with P, say $P = \overline{P_i}$. Our plan is to show via a case-by-case analysis that the last equality leads to a contradiction for each i. For that end we will need the following lemma.

Lemma 5. Assume that $\omega(\xi_1^+, \xi_2^-, \xi_3^+, \xi_4^-, \xi_5^+) \neq 0$ at a point $p \in A$, where $\xi_i^\pm$ are root vectors of ρ (as above) associated to the eigenvalues $\lambda_{\ell_i}^{\pm 1}$. Extend $\xi_i^\pm$ to smooth vector fields $\tilde{\xi}_i^\pm$ near p, tangent to the same foliations ($\mathcal{F}^u$ or $\mathcal{F}^s$) as $\xi_i^\pm$. Then, at p we must have

$$\omega(\xi_1^+, \xi_2^-, \xi_3^+, \xi_4^-, \xi_5^+) = \xi_1^+ \check{R}(\tilde{\xi}_2^-, \tilde{\xi}_3^+, \tilde{\xi}_4^-, \tilde{\xi}_5^+).$$

Proof. We have $\omega(\xi_1, \ldots, \xi_5) = (\nabla_{\xi_1}\check{R})\,(\xi_2, \ldots, \xi_5) = \xi_1\check{R}(\tilde{\xi}_2, \ldots, \tilde{\xi}_5) - \check{R}(\nabla_{\xi_1}\tilde{\xi}_2, \xi_3, \xi_4, \xi_5) - \ldots - R(\xi_2, \xi_3, \xi_4, \nabla_{\xi_1}\tilde{\xi}_5)$. Consider the term $\check{R}(\nabla_{\xi_1}\tilde{\xi}_2, \xi_3, \xi_4, \xi_5)$, and assume it is not zero. Decompose $(\nabla_{\xi_1}\tilde{\xi}_2)_p \in F_p^-$ into vectors associated to the eigenvalues $\lambda_{\ell_i}^{-1}$. There will be at least one component, say η, associated to λ^{-1}, for which $\check{R}(\eta, \xi_3, \xi_4, \xi_5) \neq 0$.

Since $\check{R}$ is Γ-invariant, we must have $|\lambda\lambda_{\ell_4}| = |\lambda_{\ell_3}\lambda_{\ell_5}|$. Combining it with $|\lambda_{\ell_2}\lambda_{\ell_4}| = |\lambda_{\ell_1}\lambda_{\ell_3}\lambda_{\ell_5}|$ we get $|\lambda_{\ell_2}| = |\lambda\lambda_{\ell_1}|$. But this is impossible, due to the pinching of K. Therefore $\check{R}(\nabla_{\xi_1} \widetilde{\xi}_2, \xi_3, \xi_4, \xi_5) = 0$. The other terms of this type vanish for the same reason. $\square$

We now proceed with an analysis similar to that of the proof of Lemma 2. The reader may wish to refer to the table of Lemma 1 and the diagram at the end of Section 3 as we go on.

1) Suppose $P = \overline{P_1}$, which corresponds to the relation $|\lambda_1^2 \lambda_2 \lambda_3^{-2}| = 1$, $|\lambda_1| < |\lambda_2| < |\lambda_3|$. One should keep in mind that $P_1 \cap P_3$ and $P_1 \cap P_4$ are not necessarily empty. Define $L(p) = \{\eta \in F_p^+ : \omega_p(\eta, \eta_1^-, \eta_2^+, \eta_3^-, \eta_4^+) = 0$, for arbitrary vectors $\eta_i^\pm \in F_p^\pm\} = \{\eta \in F_p^+ : \omega_p(\eta, F^-, F^+, F^-, F^+) = 0\}$, $p \in P$. If $\xi = a\xi_1^+ + b\xi_2^+ + c\xi_3^+ \in L(p')$ for $p' \in P_1$ and ξ_i^+ a λ_i-root vector, then $0 = \omega(\xi, F^-, F^+, F^-, F^+) = a\omega(\xi_1^+, F^-, F^+, F^-, F^+) + b\omega(\xi_2^+, F^-, F^+, F^-, F^+)$, since ω vanishes when its first argument is the fastest expanding vector. By hypothesis, for $p' \in P_1$ we have $\omega(\xi_1^+, \xi_3^-, \xi_1^+, \xi_3^-, \xi_2^+) \neq 0$ while $\omega(\xi_2^+, \xi_3^-, \xi_1^+, \xi_3^-, \xi_2^+) = 0$, even if relations 3) or 4) happen to occur at p' (see the table of relations). Hence $a = 0$. Now, $\omega(\xi_2^+, \xi_1^-, \xi_1^+, \xi_3^-, \xi_1^+) = \omega(\xi_1^+, \xi_3^-, \xi_1^+, \xi_3^-, \xi_2^+) \neq 0$ and $\omega(\xi_1^+, \xi_3^-, \xi_1^+, \xi_3^-, \xi_1^+) = 0$ (relations 1) and 5) cannot occur simultaneously at a same point), so $b = 0$ as well. On the other hand, ξ_3^+ is always in $L(p')$. Therefore $L(p')$ coincides with the fast expanding direction ξ_3^+ at points p' in P_1.

The relation $\omega_p(x, F^-, F^+, F^-, F^+) = 0$ defines a system of homogeneous algebraic equations smoothly parametrized by $p \in P$, with the property that its zero set, $L(p)$, coincides with the λ_3-direction at points in the dense set P_1. Hence $p \to L(p)$ defines a smooth line field on an open and dense subset of P containing P_1. In this way we have extended the fast expanding direction ξ_3^+ to a smooth vector field, $\tilde{\xi}_3^+$, on an open and dense subset $\mathcal{A} \subset P$ such that $P_1 \subset \mathcal{A}$. By considering $\mathcal{A} \cap J\mathcal{A}$ if necessary we may assume that $\mathcal{A}$ is J-invariant. Thus $\tilde{\xi}_3^- := J_*\tilde{\xi}_3^+$ defines an extension of ξ_3^- to a smooth vector field on $\mathcal{A}$.

Consider now $L'(p) = \{\eta \in F_p^+ : \Omega(\eta, \tilde{\xi}_3^-(p)) = 0\}$ for $p \in \mathcal{A}$, the skew-orthogonal complement of $\tilde{\xi}_3^-$ with respect to the symplectic form. $L'(p)$ defines a smooth plane field on $\mathcal{A}$ which extends the slow expanding plane span $\{\xi_1^+, \xi_2^+\}$. Denote by $\tilde{\xi}_1^+, \tilde{\xi}_2^+$ vector fields tangent to $L'(p)$ which extend ξ_1^+, ξ_2^+ respectively (they are not necessarily eigenvectors). By the Γ-invariance of $\check{R}$ we must have that $\check{R}(\tilde{\xi}_3^-, \tilde{\xi}_1^+, \tilde{\xi}_3^-, \tilde{\xi}_2^+)$ vanishes identically on $\mathcal{A}$. According to the lemma, $\omega(\xi_1^+, \xi_3^-, \xi_1^+, \xi_3^-, \xi_2^+) = 0$ on P_1. But this is a contradiction.

A similar analysis has to be carried out for the remaining cases 2), 3), 4), 5), and 6). Below we indicate the necessary changes in the argument.

2) Similar to 1)

3) Consider $L(p) = \{\eta \in F_p^- : \omega(F_p^+, \eta, F_p^+, F_p^-, F_p^+) = 0\}$

4) Consider $L(p) = \{\eta \in F_p^+ : \omega(\eta, F_p^-, \eta, F_p^-, \eta) = 0\}$, a system of cubic homogeneous equations, and show that for $p' \in P_4$, $L(p')$ coincides with the plane generated by the λ_2, λ_3-eigenspaces. Then proceed as in case 1).

6) Consider $L(p) = \{\eta \in F_p^- : \omega(F^+, \eta, F^+, \eta, F^+) = 0\}$, a system of quadratic equations, and show that for $p' \in P_6$, $L(p')$ coincides with the λ_1, λ_2-plane. Then proceed as before.

5) We may use a somewhat different argument, here. Assume $P = \overline{P_5}$, which corresponds to the relation $|\lambda_1^3| = |\lambda_3^2|$. Note that no other relation may occur simultaneously. Since $\dim F^{\pm} = 3$, either F_1^+ or F_3^+ is one-dimensional.

a) Suppose F_1^+ is one-dimensional. Let $\xi_1, \xi_3, \xi_5 \in F_1^+$, $\xi_2, \xi_4 \in F_3^-$ be vectors at $p' \in P_5$ such that $\omega(\xi_1, \ldots, \xi_5) \neq 0$. Extend ξ_i to a smooth vector field $\tilde{\xi}_i$ tangent to the same bundle as $\xi_i(F^+$ or $F^-)$. Consider $p \to L(p) = \ker \omega(\cdot, \tilde{\xi}_2, \tilde{\xi}_3, \tilde{\xi}_4, \tilde{\xi}_5)\big|_{F^+}$ for p close enough to p' so that $\omega_p \neq 0$. This function defines a smooth filed of planes containing F_2^+ and F_3^+ at points of P_5. From here on, proceed as in the previous cases.

b) Suppose F_3^+ is one-dimensional. Use a similar argument, but now starting with $L(p) = \ker \omega(\tilde{\xi}_1, \cdot, \tilde{\xi}_3, \tilde{\xi}_4, \tilde{\xi}_5)\big|_{F^-}$. □

6. Concluding Remarks.

In the two-dimensional case a similar rigidity result was proved by E. Ghys [2]. Kanai's method also works in that case without any pinching assumption and produces a somewhat more natural proof. In fact, a much weaker smoothness assumption forces rigidity in dimension two [3]. Moreover, in the same paper S. Hurder and the second author show that higher than usual smoothness of the stable foliation (and hence rigidity) is equivalent to vanishing of a certain 1-cocycle over the geodesic flow, which is called Anosov cocycle to emphasize the fact that in Chapter 24 of his fundamental work [1] Anosov discovered obstructions to smoothness which turned out to be values of that cocycle. It would be very interesting to understand the nature of critical smoothness in higher-dimensional situation. It looks that at least partially the situation can be described in cocycle terms. Unlike dimension two, however, the "critical smoothness" should depend on relations betwen the Lyapunov exponents at periodic points.

In the two-dimensional case rigidity of the geodesic flow implies that the metric itself has constant curvature. This follows from entropy rigidity [5] or rigidity of Godbillon-Vey class [3]. Since none of the two facts is known in higher dimension the question remains open.

Our methods allow to shed some light on the situation in dimension greater than four. Results for that case will appear in a separate paper.

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