Harmonic functions over group actions

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To Bob Zimmer, on his 60th birthday

Abstract

Let X be a compact space, Γ a countable group of homeomorphisms of X, and μ a probability measure on Γ. From this data one naturally defines a random walk on X and a related notion of continuous harmonic function. The question we wish to investigate is whether all continuous harmonic functions on X are Γ-invariant. If this is the case we say that the system satisfies the topological *Liouville property*. We show that if μ is a symmetric probability measure on an arbitrary Γ and $X = S^1$ or the interval $[0, 1]$, then the Liouville property always holds. In addition, when the Poisson boundary of (Γ, μ) can be identified with the circle, we give a general construction of a non-Liouville action of Γ on S^2, generalizing an example from [FZ2].

1 Introduction

A useful idea in the study of group actions without invariant measures is to introduce and study a random $\mathbb{Z}$-action associated to a choice of probability measure on the acting group. To fix notations, let Γ denote a group of transformations of a space X and μ a probability measure on Γ. Then μ defines a random walk on Γ, which induces a random $\mathbb{Z}$-action on X. See, for example, [Fu] where this point of view is well-represented. It is often assumed in this context that the Γ-space admits a μ-stationary measure (also called μ-harmonic measure—see definition below) of full support, endowing the random $\mathbb{Z}$-action with an invariant measure. This brings along the ergodic theory apparatus that relies on the existence of invariant measures.

Often, however, the Γ-space has large regions with possibly interesting dynamics that are not detected by any harmonic measure for a given μ. If this is the case, one can hope to gain some insight into the system represented by (X, Γ, μ) by investigating the space of μ-harmonic functions on X. As pointed out in Proposition 2.4 below, continuous μ-harmonic functions on compact Γ-spaces are Γ-invariant on the support of any μ-harmonic probability measure, so non-trivial (i.e., non-Γ-invariant) such functions may yield information about the regions that harmonic measures do not detect. This is the perspective motivating this paper. We have chosen here to consider compact Γ-spaces and continuous functions, although one may also try to pursue the subject on measurable Γ-spaces.

The most basic question to ask about (X, Γ, μ) in the present context is whether it satisfies a topological *Liouville property*, i.e., whether all continuous μ-harmonic functions are Γ-invariant. The main result of the paper is that the Liouville property holds for actions of countable groups on one-dimensional spaces when μ is a symmetric probability measure.

Theorem 1.1 *Let Γ be a countable group acting on X by homeomorphisms, where X is either the circle S^1 or the interval $[0, 1]$. Let μ be a symmetric probability measure on Γ. Then the Liouville property holds for (X, Γ, μ).*

We give at the end of the paper an example of Γ-space for which the Liouville property does not hold. More specifically, we show that if the Poisson boundary of (Γ, μ) can be identified with the circle S^1, then it is possible to construct a Γ-action on the two-sphere so that (S^2, Γ, μ) admits continuous μ-harmonic functions that are non-constant on all orbits in the complement of a pair of fixed points. The action can be chosen to be ergodic with respect to the smooth measure class on S^2, whereas the random walk is transient in the complement of the two fixed points. A general recipe for constructing non-Liouville actions is also proposed.

The proof of Theorem 1.1 parallels the proof of the main theorem of [FFP]. That paper is about the Liouville property for codimension-one foliated Brownian motion and harmonic functions for the Laplace-Beltrami operator associated to a Riemannian metric on leaves, rather than discrete group actions. Although the two situations cannot be directly compared, our assumptions on Γ and μ are, in a sense, much more general than related assumptions on that paper. The example of a non-Liouville action on S^2 generalizes a similar construction given in [FZ2] in the context of foliations with a Riemannian metric on leaves. See, also, [FZ1] for these issues in the setting of holomorphic functions on foliations. The paper [DK] has some deep results about harmonic functions on foliated spaces in the spirit of the present discussion.

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2 Harmonic functions on Γ-spaces

In this section we collect some basic properties and general remarks concerning harmonic functions.

Let Γ be a countable (infinite) group, μ a probability measure on Γ, and X a compact topological space on which Γ acts by homeomorphisms. It will always be assumed that μ is *non-degenerate*, i.e., the support of μ generates Γ as a semigroup. In particular, $\mu(\gamma) < 1$ for all $\gamma \in \Gamma$. A continuous function $f : X \to \mathbb{R}$ is said to be *μ-harmonic* if, for all $x \in X$,

$$f(x) = \sum_{\gamma\in\Gamma} f(\gamma(x))\mu(\gamma).$$

Let $H(X, \Gamma, \mu)$ denote the set of μ-harmonic functions on X and $C(X)^\Gamma$ the subset consisting of continuous Γ-invariant functions. If $H(X, \Gamma, \mu) = C(X)^\Gamma$, we say that (X, Γ, μ) satisfies the *Liouville property*.

We note that if $f \in H(X, \Gamma, \mu)$ and $x \in X$, then $\tilde{f}_x(\gamma) := f(\gamma^{-1}(x))$ is a harmonic function on Γ in the following sense:

$$\tilde{f}_x(\gamma) = \sum_{\eta\in\Gamma} \tilde{f}_x(\gamma\eta)\mu(\eta^{-1}).$$

Under the conventions of [Fu], $\tilde{f}_x$ is harmonic relative to the right $\hat{\mu}$-random walk on Γ, where $\hat{\mu}(\gamma) = \mu(\gamma^{-1})$. Thus, if $(\Gamma, \hat{\mu})$ itself has the Liouville property, i.e., if bounded harmonic functions on Γ are constant, then necessarily (X, Γ, μ) has the Liouville property for all X. Therefore, the present subject is uninteresting if the Poisson boundary of $(\Gamma, \hat{\mu})$ reduces to a single point. It is known, for example, that if Γ is amenable, then there exists a symmetric $(\mu = \hat{\mu})$ non-degenerate μ for which the Poisson boundary is trivial.

A good example of group to keep in mind, on the other hand, is the fundamental group of a closed surface of genus at least 2, represented as isometries of the Poincaré disc. It has a nontrivial boundary for many interesting probability measures. For example, a result due to Furstenberg implies that on this group there exists a probability measure μ relative to which the boundary of Γ coincides with the geometric boundary, S^1, of the disc [Fur].

Let $\Delta = P - I$ be the *Laplace operator* on the set $C(X)$ of real valued continuous functions on X, where $(Pf)(x) := \sum \mu(\gamma) f(\gamma(x))$ is the averaging operator and I is the identity. Clearly, f is harmonic iff $\Delta f = 0$. More generally, f satisfying $\Delta f \geq 0$ (resp., $\Delta f \leq 0$) is said to be *superharmonic* (resp., *subharmonic*).

Proposition 2.1 (Maximum principle) *If a superharmonic $f \in C(X)$ is bounded above by C and $f(x) = C$ for an $x \in X$, then f is constant on the Γ-orbit of x.*

Proof. This trivially follows from μ being non-degenerate and the remark

$$0 \leq \sum_{\gamma \in \Gamma} (f(\gamma(x)) - f(x))\mu(\gamma) \leq \sum_{\gamma \in \Gamma} (f(\gamma(x)) - C)\mu(\gamma) \leq 0,$$

so $f(\gamma(x)) = C$ for all γ in the support of μ. $\square$

Corollary 2.2 *If the closure of every Γ-orbit contains a unique minimal set, then the Liouville property holds.*

Proof. Let X_0 be the closure of an orbit. The maximum and minimum values in X_0 of a harmonic function are attained on orbits where the function is constant. Since X_0 has a single minimal set, these two values must agree. $\square$

It is useful to introduce a (directed) metric d_{x_0} on the orbit Γx_0 as follows. Given x, y in Γx_0, we specify a *path* joining x to y by a sequence $\alpha = (\gamma_0, \gamma_1, \ldots, \gamma_{n-1})$, $\mu(\gamma_i) > 0$, such that $y_0 = x$, $y_n = y$, and $y_{i+1} = \gamma_i y_i$ for $i = 0, \ldots, n-1$. The *length* of α is defined as

$$L(\alpha) = \sum_{i=0}^{n-1} - \ln \mu(\gamma_i) \in (0, \infty].$$

We allow x to be joined to itself by the trivial path, defined by $n = 0$, $\alpha = \emptyset$ and $L(\alpha) = 0$. Now let $d_{x_0}(x, y) = \inf L(\alpha)$, where the infimum is taken over all n and all $(\gamma_0, \ldots, \gamma_{n-1})$ describing a path from x to y.

Proposition 2.3 (Harnack inequality) *Let f be a continuous, positive, subharmonic function on X and $x_0 \in X$ arbitrary. Then for all $x, y \in \Gamma x_0$,*

$$\frac{f(y)}{f(x)} \leq e^{d_{x_0}(x,y)}.$$

Proof. As f is subharmonic, $f(x) \geq \sum_{\gamma \in \Gamma} \mu(\gamma) f(\gamma x) \geq \mu(\eta) f(\eta x)$ for all $\eta \in \Gamma$, so $f(\eta x)/f(x) \leq e^{-\ln \mu(\eta)}$. Multiplying such inequalities along any path α from x to y yields the claim. $\square$

Given (X, Γ, μ) as above, a measure ν on X is called *harmonic* if ν is Γ-quasi-invariant and $\mu * \nu = \nu$, where the convolution $\mu * \nu$ is the image of the product measure $\mu \otimes \nu$ under the action map $\Gamma \times X \to X$. It is easily shown that $\mu * \nu = \nu P$, where the averaging operator P acts on ν by duality as

$$\nu P = \sum_{\gamma \in \Gamma} \mu(\gamma) \gamma_* \nu.$$

The following useful remark is due, in the context of foliated Brownian motion, to Garnett [Gar]. The proof, which is sketched below for completeness, is adapted from Paulin [Pa].

Proposition 2.4 *Let ν be a harmonic probability measure on (X, Γ, μ). Then every h in $H(X, \Gamma, \mu)$ is Γ-invariant on the support of ν.*

Proof. It may be assumed that h is non-negative. Let c be a non-negative constant and define $h_c(x)$ as the minimum of $\{h(x), c\}$. Note that $Ph_c \leq h_c$. Since ν is harmonic, $\int h_c d\nu = \int h_c d(\mu * \nu) = \int Ph_c d\nu$, therefore $\int (Ph_c - h_c) d\nu = 0$. As the functions involved are continuous, it follows that $Ph_c = h_c$ on the support of ν. Applying the maximum principle to $h_c|_{\mathrm{supp}(\nu)}$, we conclude that $h^{-1}([c, \infty)) \cap \mathrm{supp}(\nu)$ is Γ-invariant for all c. This implies that h is Γ-invariant on $\mathrm{supp}(\nu)$. $\square$

Let $\mathcal{C}$ denote the largest Γ-invariant subset of X such that $h|_{\mathcal{C}}$ is Γ-invariant for all $h \in H(X, \Gamma, \mu)$. Clearly, $\mathcal{C}$ is closed. It is also non-empty since, by the maximum principle, it contains every minimal set in X. Thus if $\mathcal{M}$ denotes the closure of the union of all the minimal sets of X relative to the Γ-action, then $\mathcal{M} \subseteq \mathcal{C}$. Due to Proposition 2.4, the set $\mathcal{S}$, defined as the closure of the union of supports of all harmonic measures, is also contained in $\mathcal{C}$. In fact, it is easily seen that $\mathcal{M} \subseteq \mathcal{S} \subseteq \mathcal{C}$. It will be explained shortly that the random walk associated to (X, Γ, μ) (defined next) converges to $\mathcal{C}$ with probability 1.

3 Random walk on (X, Γ, μ)

Let (X, Γ, μ) be, as above, a compact Γ-space where Γ is a countable group and μ is a probability measure on Γ. The measure induces a Markov transition kernel on X by setting $p(x, y) = 0$ if $y \notin \Gamma x$ and

$$p(x, y) = \sum_{y = \gamma x} \mu(\gamma)$$

where the sum is over all $\gamma \in \Gamma$ such that $\gamma x = y$. Let X_n denote the random walk on X with transition probabilities p and initial state x_0. For the set $\mathcal{C}$, defined at the end of the previous section, we have the following.

Proposition 3.1 *The random walk X_n on $X \setminus \mathcal{C}$ is transient.*

The proposition means that for all $x_0 \in X \setminus \mathcal{C}$ and every open U containing $\mathcal{C}$, there exists a random integer N, almost surely finite, such that $X_n \in U$ for all $n \geq N$. Prior to proving this fact we need to review some information about the Poisson boundary of Γ. Our main source is Furman [Fu].

The boundary of Γ can be described as a compact Hausdorff Γ-space B endowed with a probability measure of full support, ν, satisfying the following properties:

1. ν is harmonic relative to $(\Gamma, \hat{\mu})$;

2. For almost every sample path $\bar{\gamma} = (\gamma_1, \gamma_2, \dots)$ of a (right) $\hat{\mu}$-random walk on Γ, a limit probability $\nu_{\bar{\gamma}} = \lim_{n \to \infty} (\gamma_1 \dots \gamma_n)_* \nu$ exists and is equal to a Dirac measure δ_{γ_∞} supported on a random point, η_∞, of B;

3. Let $H^\infty(\Gamma, \hat{\mu})$ denote the space of bounded harmonic functions on Γ relative to $\hat{\mu}$. Then the map $F : L^\infty(B, \nu) \to H^\infty(\Gamma, \hat{\mu})$ given by $F(\phi)(\gamma) = \int_B \phi \, d\gamma_* \nu$ is an isometric bijection such that, for $h = F(\phi)$,
$$\phi(\eta_\infty) = \lim_{n \to \infty} h(\gamma_1 \dots \gamma_n).$$

It is observed in [Fu], Remark 2.16 (i), that if Γ is a discrete group then, for the topological description of B given there (denoted $\overline{B}$ in Furman's article), $L^\infty(B, \nu)$ coincides with the continuous functions on B. In this case, B is non-metrizable if Γ has non-trivial Poisson boundary. However, measure theoretically the Poisson boundary is always a Lebesgue space.

Lemma 3.2 *Let $\eta_n = \gamma_1 \dots \gamma_n$ be a right $\hat{\mu}$-random walk starting at the unit element $e \in \Gamma$. Let $h \in H^\infty(\Gamma, \hat{\mu})$ and C a positive constant. Then, for a.e. sample path η_n and every sequence $\xi_n \in \Gamma$ such that $d_e(\eta_n, \xi_n) < C$, the sequences $h(\eta_n)$ and $h(\xi_n)$ converge to the same value. (Here d_e is the directed metric used in Proposition 2.3.)*

Proof. Define $\nu_\gamma = \gamma_* \nu$. From the above properties of the Poisson boundary and Proposition 2.3, it follows that

$$\max\left\{\frac{d\nu_{\eta_n}}{d\nu_{\xi_n}}, \frac{d\nu_{\xi_n}}{d\nu_{\eta_n}}\right\} \leq e^C$$

for all n. But since ν_{η_n} converges to the Dirac measure δ_{η_∞}, the sequence ν_{ξ_n} must converge to the same measure. $\square$

We now return to Proposition 3.1. Let $h \in H(X, \Gamma, \mu)$. As pointed out above, for each $x \in X \setminus \mathcal{C}$ we obtain a harmonic function $\tilde{h}_x \in H^\infty(\Gamma, \hat{\mu})$. By Lemma 3.2, for each $C > 0$, for a.e. sample path $(x_0, x_1, \dots)$ of the random walk on X, and for every sequence $(y_0, y_1, \dots)$ such that $d_x(x_i, y_i) < C$ for all i, both sequences $\tilde{h}_x(x_i)$ and $\tilde{h}_x(y_i)$ converge and have the same limit. Therefore, as C is arbitrary and h is continuous, h must be constant on the orbit of every limit point of x_i. But this is to say that all limit points must lie in $\mathcal{C}$. This concludes the proof of Proposition 3.1.

One further simple remark, concerning induced random walks for subgroups, will be needed. Let Γ_0 be a finite index subgroup of Γ. Let N^γ denote the first time, $n \geq 1$, at which random walk on $(\Gamma, \hat{\mu})$ starting at γ reaches Γ_0. Then N^γ is a Markov time, almost surely finite. Now define a probability measure on Γ_0 by $\hat{\mu}_0(\eta) = \mathrm{Prob}(\gamma_{N^e} = \eta)$, that is, $\hat{\mu}_0(\eta)$ is the probability that random walk, $(\gamma_0, \gamma_1, \dots)$, on Γ, starting at the unit element, will first return to Γ_0 at η. General properties of Markov times and martingales imply that the restriction to Γ_0 of a function in $H^\infty(\Gamma, \hat{\mu})$ is harmonic relative to $\hat{\mu}_0$.

Lemma 3.3 *Let Γ_0 have finite index in Γ and μ_0 the image of the induced measure $\hat{\mu}_0$ under group inverse. Then $H(X, \Gamma, \mu) \subseteq H(X, \Gamma_0, \mu_0)$. In particular, if (X, Γ_0, μ_0) satisfies the Liouville property, then so does (X, Γ, μ).*

Proof. This is due to the above remarks about μ_0 and the relationship between a harmonic functions h on (X, Γ, μ) and $\tilde{h}_x$ on $(\Gamma, \hat{\mu})$ or $(\Gamma_0, \hat{\mu}_0)$. (See the beginning of section 2.) $\square$

4 Actions on the circle

We now consider actions of discrete (countable) groups on the circle. The following lemma is a well-known group-actions version of Poincaré's classification of circle homeomorphisms.

Lemma 4.1 *Let Γ be a countable group of homeomorphisms of S^1. Then one of the following holds:*

1. *The action is minimal;*
2. *The action is not minimal and there is a unique minimal set;*
3. *There is a finite orbit.*

Proof. It suffices to prove the lemma for a subgroup of finite index. Thus we may assume that Γ is orientation preserving. Further assume that the action is not minimal and contains no finite orbit. Let Z be a minimal set and U an arbitrary connected component of $S^1 \setminus Z$. This minimal set is unique if the orbit of every point in U can be shown to accumulate on Z. Now, U and γU either are disjoint or agree for each $\gamma \in \Gamma$. If the latter, then the end points of U are fixed by γ. As the subgroup of Γ fixing the end points of U has infinite index, there are infinitely many disjoint intervals of the form γU, so we can choose a sequence $\gamma_n U$ decreasing to 0 in length. Therefore, the orbit of every point in U must limit on Z. □

By Corolary 2.2, having a unique minimal set implies the Liouville property. Thus, in order to prove Theorem 1.1, it suffices to assume that the action on S^1 contains a finite orbit. By Lemma 3.3, it can be assumed that Γ has a fixed point in S^1 and preserves orientation. This reduces the proof of Theorem 1.1 to showing the Liouville property for orientation preserving actions on the interval $[0, 1]$.

5 Actions on the interval

We now restrict attention to systems $([0, 1], \Gamma, \mu)$ for which the Γ-action is orientation preserving. Clearly, then, the only finite orbits are fixed points, and since the union of the fixed points is a closed subset of $[0, 1]$, we may further restrict attention to intervals without interior finite orbits.

Lemma 5.1 *If Γ acts by orientation preserving homeomorphisms of $[0, 1]$ without interior finite orbits, the orbit of every $x \in (0, 1)$ must limit on both 0 and 1.*

Proof. The supremum and infimum of Γx are easily seen to be fixed points. □

Lemma 5.2 *Suppose that the Liouville property does not hold for $([0, 1], \Gamma, \mu)$ and the Γ-action is orientation preserving without interior finite orbits. Then:*

1. *There is a unique continuous harmonic f not Γ-invariant such that $f(0) = 0$, $f(1) = 1$;*
2. *$f(x)$ is the probability that random walk on $([0, 1], \Gamma, \mu)$ starting at x converges to 1;*
3. *f is not constant on any interior orbit;*
4. *f is increasing.*

Proof. Let g be a continuous, harmonic, non-Γ-invariant function on $[0, 1]$. By Lemma 5.1 and the maximum principle, $g(0) \neq g(1)$, and $g(x)$ lies in the open interval with end points $g(0), g(1)$ for each $x \in (0, 1)$. Lemma 5.1 also implies that g cannot be constant on any interior orbit, so the set $\mathcal{C}$ of Proposition 3.1 coincides with $\{0, 1\}$. By composing g with an appropriate affine function of $\mathbb{R}$, one obtains f such that $f(i) = i$ for $i = 0, 1$ and $0 < f(x) < 1$ on interior points. Uniqueness is due to the maximum principle. By Proposition 3.1, the random walk X_n^x on $[0, 1]$ starting at x must converge to a random point $X_\infty \in \{0, 1\}$ with probability 1. As f is harmonic, the expected value $E[f(X_n^x)]$ is equal to $f(x)$ and $\lim_{n\to\infty} f(X_n)$ exists almost surely. By continuity, $f(X_n^x)$ converges to either 0 or 1. Therefore,

$$f(x) = \lim_{n\to\infty} E[f(X_n^x)] = f(0)\text{Prob}(X_n^x \to 0) + f(1)\text{Prob}(X_n^x \to 1) = \text{Prob}(X_n^x \to 1).$$

Finally, given any $x_1 < x_2$ and a sample path $(\gamma_1, \gamma_2, \dots)$ on Γ for the right $\hat{\mu}$-random walk, the corresponding sample paths for the random walks on $[0, 1]$ satisfy $X_n^{x_1} < X_n^{x_2}$. Therefore, the probability that $X_n^{x_2}$ converges to 1 is at least as great as the probability that $X_n^{x_1}$ converges to 1. This shows that $f(x)$ is increasing. □

We can now show how the assumption of a non-Γ-invariant harmonic function on $[0,1]$ leads to a contradiction. Let f be as in Lemma 5.2 and define a probability measure ν on $[0,1]$ by extending the definition

$$\nu((a,b]) = f(b) - f(a)$$

to the Lebesgue measurable subsets of of the interval. At this point we make the further assumption that the measure μ on Γ is symmetric. Then,

$$\begin{aligned}
\mu * \nu((a,b]) &= \sum_{\gamma\in\Gamma} \mu(\gamma)\nu\left((\gamma^{-1}a, \gamma^{-1}b]\right) \\
&= \sum_{\gamma\in\Gamma} \mu(\gamma)\left(f(\gamma^{-1}b) - f(\gamma^{-1}a)\right) \\
&= \sum_{\gamma\in\Gamma} \mu(\gamma) f(\gamma b) - \sum_{\gamma\in\Gamma} \mu(\gamma) f(\gamma a) \\
&= f(b) - f(a) \\
&= \nu((a,b]).
\end{aligned}$$

This remark, which was shown to us by B. Deroin, simplifies a similar but somewhat more involved argument of an earlier version of this paper. The same argument is used in the proof of Proposition 5.7 of [DKN]. From this we obtain the following lemma.

Lemma 5.3 *If μ is symmetric, ν is a harmonic probability measure.*

We can now conclude the proof of Theorem 1.1. Since by Proposition 2.4 any function in $H(X,\Gamma,\mu)$ must be Γ-invariant on the support of a harmonic probability measure, and since the above f is not constant on any interior orbit of the Γ-action on $[0,1]$, we arrive at a contradiction. Therefore, a non-Γ-invariant, continuous harmonic function cannot exist.

6 Examples

We give now a class of examples of Γ-spaces to illustrate the way in which the Liouville property can fail to hold. We begin with a general remark that suggests a recipe for constructing examples. Let μ be a probability measure on the countable (discrete) group Γ and let $\hat{\mu}$, as before, be the image of μ under group inverse. The unit ball in $H^\infty(\Gamma,\hat{\mu})$ (the latter equipped with the supremum norm) is a compact space, which we denote by X_0. Note that Γ acts on X_0 by homeomorphisms under the definition $(\gamma,\phi)\mapsto \gamma\cdot\phi$, where $(\gamma\cdot\phi)(\eta) = \phi(\gamma^{-1}\eta)$.

The Γ-space X_0 has a tautological continuous μ-harmonic function: $f(\phi) = \phi(e)$. This is, in fact, μ-harmonic since

$$\sum_{\gamma\in\Gamma} f(\gamma\cdot\phi)\mu(\gamma) = \sum_{\gamma\in\Gamma}\phi(\gamma^{-1})\mu(\gamma) = \sum_{\gamma\in\Gamma}\phi(\gamma)\hat{\mu}(\gamma) = \phi(e) = f(\phi).$$

Furthermore, since $f(\gamma\phi) = \phi(\gamma^{-1})$, f is not constant on $\Gamma\cdot\phi$ if ϕ itself is not a constant function. This simple remark suggests the following approach to finding examples of non-Liouville actions on a Γ-space S. Suppose we can somehow construct a continuous, Γ-equivariant map $\Phi : S \to X_0$. We express equivariance by $\Phi(\gamma s) = \gamma\Phi(s)$ and write $\Phi(s) = \phi_s$. If Φ does not map S entirely into the space of constant functions, then $f\circ\Phi$ defines a continuous, μ-harmonic function on S which is not Γ-invariant. Thus, what we have shown in Theorem 1.1 amounts to the following.

Proposition 6.1 *If μ is a symmetric probability measure on Γ, then every Γ-equivariant continuous $\Phi : S^1 \to X_0$ maps into the constant functions.*

We now construct a continuous Γ-equivariant map $\Phi : S^2 \to X_0$ whose image is not contained in the space of constant functions. By the above remark, this yields a non-Liouville action on S^2. The example generalizes one given in [FZ2].

Let Γ and $\hat{\mu}$ be such that the Poisson boundary of $(\Gamma, \hat{\mu})$ can be identified with the circle S^1. For example, as already noted, if Γ is a uniform group of isometries of the Poincaré disc, it is possible to find a symmetric μ for which the Poisson boundary of the group coincides with the geometric boundary of the disc [Fur]. Let ν be a harmonic measure on S^1 as in the discussion immediately after Proposition 3.1. The following general fact was pointed out to us by Kaimanovich.

Lemma 6.2 *Let $\hat{\mu}$ be a non-degenerate probability measure on Γ such that a Poisson boundary (B, ν) of $(\Gamma, \hat{\mu})$ is non-trivial. Then the $\hat{\mu}$-harmonic measure ν has no atoms.*

Proof. Suppose for a contradiction that ν does have atoms, and let $b \in B$ be such that $\nu(b) \geq \nu(b')$ for all $b' \in B$. Since ν is $\hat{\mu}$-harmonic and $\hat{\mu}$ is non-degenerate, the maximum principle applied to $\gamma \mapsto (\gamma_* \nu)(b)$ implies $\nu(b) = \nu(\gamma b)$ for all $\gamma \in \Gamma$. On the other hand, if $(\gamma_1, \gamma_2, \dots)$ is a random walk on $(\Gamma, \hat{\mu})$, then $(\gamma_1 \dots \gamma_n)_* \nu$ converges to the Dirac measure δ_{g_∞}, where g_∞ is the random point on the boundary to which the random walk converges. Writing $g_n = \gamma_1 \dots \gamma_n$, then $0 < \nu(b) = (g_n)_* \nu(b) \to \delta_{g_\infty}(b)$. This is only possible if sample paths converge to b almost surely, contradicting the assumption that the Poisson boundary is non-trivial. $\square$

We now construct a non-trivial Γ-equivariant continuous map $\Phi : S^2 \to X_0$. Let $x = (z, \theta)$ in the cylinder $S^1 \times [0, 2\pi]$ represent the interval $I_x \subset S^1$ with endpoints z and $ze^{i\theta}$. Naturally, $\gamma I_x = I_{\gamma x}$. The circle boundaries $S^1 \times \{0\}$ and $S^1 \times \{2\pi\}$ are invariant under Γ and correspond to intervals of length 0 or 2π. The quotient $(S^1 \times [0, 2\pi])/\sim$ obtained by collapsing the boundary circles to points is homeomorphic to S^2 and the Γ-action on the cylinder defines an action on the quotient by homeomorphisms fixing two points, denoted $N = (S^1 \times \{2\pi\})/\sim$ and $S = (S^1 \times \{0\})/\sim$. This defines S^2 as a topological Γ-space.

Let ν be a harmonic probability measure on S^1, which is granted since S^1 is a Poisson boundary of $(\Gamma, \hat{\mu})$. Define $\Phi(x) = \phi_x$, where $\phi_x(\gamma) = \gamma_* \nu(I_x)$. Note that Φ is well-defined since $\nu(I_x)$ is 0 on all intervals of length 0, and 1 on all intervals of length 2π. A simple calculation shows that ϕ_x is $\hat{\mu}$-harmonic for each $x \in S^2$ and that Φ is Γ-equivariant. Furthermore, Φ is continuous due to the fact that ν has no atoms. Since the Poisson boundary is non-trivial, Φ does not map into the set of constant functions. This shows that (S^2, Γ, μ) is non-Liouville.

In this example, the only Γ-orbits in S^2 on which $\Phi(x)$ is a constant function are the fixed points N, S, so $\mathcal{C} = \{N, S\}$, where $\mathcal{C}$ is the set defined in Proposition 3.1. The same argument used in Lemma 5.2 shows that $g = f \circ \Phi$ can be composed with an affine transformation of $\mathbb{R}$ to insure $g(S) = 0$ and $g(N) = 1$, after which $g(x)$ is the probability that random walk on (S^2, Γ, μ) starting at $x \in S^2$ converges to N.

When (S^1, ν) coincides with the boundary of the hyperbolic disc, where Γ is a uniform lattice in $PSL(2, \mathbb{R})$, Lemma 4.2 of [FZ2] can be used to show that the just constructed action on S^2 is ergodic relative to the smooth measure class on S^2. This dynamical property of the action contrasts with the simple behavior of the random walk, which converges to N or S with probability 1.

It is interesting to regard the induced random walk on the space X_0 itself, for a general $(\Gamma, \hat{\mu})$. Let $I = [-1, 1] \subset X_0$ represent the subspace of constant functions in X_0. Then the random walk on (X_0, Γ, μ) converges towards I, i.e., the random walk on $X_0 \setminus I$ is transient. On the other hand, the Γ-action on X_0 can be dynamically very complicated. For example, it is shown in [FZ2] for subgroups of $PSL(2, \mathbb{R})$ that the $\mathbb{Z}$-action on X_0 induced by a parabolic or hyperbolic element is a chaotic dynamical system.

References

[DK] B. Deroin and V. Kleptsyn. *Random conformal dynamical systems*, GAFA, Vol. 17 (2007) 1043-1105.

[DKN] B. Deroin, V. Klepstyn, A. Navas. *Sur la dynamique unidimensionnelle en régularité intermédiaire*, preprint, 2006.

[FFP] S. Fenley, R. Feres, K. Parwani. *Harmonic functions on* $\mathbb{R}$*-covered foliations*, 2008 (to appear).

[FZ1] R. Feres and A. Zhegib. *Leafwise holomorphic functions*, Proc. Amer. Math. Soc. 131 (2003), no. 6, 1717–1725.

[FZ2] R. Feres and A. Zhegib. *Dynamics on the space of harmonic functions and the foliated Liouville problem*, Ergodic Theory Dynam. Systems 25 (2005), no. 2, 503–516.

[Fu] A. Furman. *Random Walks on Groups and Random Transformations*, Handbook of Dynamical Systems, Vol. 1A, Chapter 12. Hasselblatt and Katok, North-Holland, 2002, 931-1014.

[Fur] H. Furstenberg. *Random walks and discrete subgroups of Lie groups*, 1971 Advances in Probability and Related Topics, Vol. 1 pp. 1-63, Dekker, New York.

[Gar] L. Garnett. *Foliations, the ergodic theorem and Brownian motion*, J. Funct. Anal. 51 (1983), 285-311.

[Pa] F. Paulin. *Analyse harmonique des relations d'équivalence mesurées discrètes*, Markov Proc. Rel. Fields. 5 (1999) 163-200.

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