Harmonic functions over group actions

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To Bob Zimmer, on his 60th birthday

Abstract

Let $X$ be a compact space, $\Gamma$ a countable group of homeomorphisms of $X$, and $\mu$ a probability measure on $\Gamma$. From this data one naturally defines a random walk on $X$ and a related notion of continuous harmonic function. The question we wish to investigate is whether all continuous harmonic functions on $X$ are $\Gamma$-invariant. If this is the case we say that the system satisfies the topological Liouville property. We show that if $\mu$ is a symmetric probability measure on an arbitrary $\Gamma$ and $X = S^1$ or the interval $[0, 1]$, then the Liouville property always holds. In addition, when the Poisson boundary of $(\Gamma, \mu)$ can be identified with the circle, we give a general construction of a non-Liouville action of $\Gamma$ on $S^2$, generalizing an example from [FZ2].

1 Introduction

A useful idea in the study of group actions without invariant measures is to introduce and study a random $\mathbb{Z}$-action associated to a choice of probability measure on the acting group. To fix notations, let $\Gamma$ denote a group of transformations of a space $X$ and $\mu$ a probability measure on $\Gamma$. Then $\mu$ defines a random walk on $\Gamma$, which induces a random $\mathbb{Z}$-action on $X$. See, for example, [Fu] where this point of view is well-represented. It is often assumed in this context that the $\Gamma$-space admits a $\mu$-stationary measure (also called $\mu$-harmonic measure—see definition below) of full support, endowing the random $\mathbb{Z}$-action with an invariant measure. This brings along the ergodic theory apparatus that relies on the existence of invariant measures.

Often, however, the $\Gamma$-space has large regions with possibly interesting dynamics that are not detected by any harmonic measure for a given $\mu$. If this is the case, one can hope to gain some insight into the system represented by $(X, \Gamma, \mu)$ by investigating the space of $\mu$-harmonic functions on $X$. As pointed out in Proposition 2.4 below, continuous $\mu$-harmonic functions on compact $\Gamma$-spaces are $\Gamma$-invariant on the support of any $\mu$-harmonic probability measure, so non-trivial (i.e., non-$\Gamma$-invariant) such functions may yield information about the regions that harmonic measures do not detect. This is the perspective motivating this paper. We have chosen here to consider compact $\Gamma$-spaces and continuous functions, although one may also try to pursue the subject on measurable $\Gamma$-spaces.

The most basic question to ask about $(X, \Gamma, \mu)$ in the present context is whether it satisfies a topological Liouville property, i.e., whether all continuous $\mu$-harmonic functions are $\Gamma$-invariant. The main result of the paper is that the Liouville property holds for actions of countable groups on one-dimensional spaces when $\mu$ is a symmetric probability measure.

**Theorem 1.1** Let $\Gamma$ be a countable group acting on $X$ by homeomorphisms, where $X$ is either the circle $S^1$ or the interval $[0, 1]$. Let $\mu$ be a symmetric probability measure on $\Gamma$. Then the Liouville property holds for $(X, \Gamma, \mu)$.

We give at the end of the paper an example of $\Gamma$-space for which the Liouville property does not hold. More specifically, we show that if the Poisson boundary of $(\Gamma, \mu)$ can be identified with...
the circle $S^1$, then it is possible to construct a $\Gamma$-action on the two-sphere so that $(S^2, \Gamma, \mu)$ admits continuous $\mu$-harmonic functions that are non-constant on all orbits in the complement of a pair of fixed points. The action can be chosen to be ergodic with respect to the smooth measure class on $S^2$, whereas the random walk is transient in the complement of the two fixed points. A general recipe for constructing non-Liouville actions is also proposed.

The proof of Theorem 1.1 parallels the proof of the main theorem of [FFP]. That paper is about the Liouville property for codimension-one foliated Brownian motion and harmonic functions for the Laplace-Beltrami operator associated to a Riemannian metric on leaves, rather than discrete group actions. Although the two situations cannot be directly compared, our assumptions on $\Gamma$ and $\mu$ are, in a sense, much more general than related assumptions on that paper. The example of a non-Liouville action on $S^2$ generalizes a similar construction given in [FZ2] in the context of foliations with a Riemannian metric on leaves. See, also, [FZ1] for these issues in the setting of holomorphic functions on foliations. The paper [DK] has some deep results about harmonic functions on foliated spaces in the spirit of the present discussion.

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2 Harmonic functions on $\Gamma$-spaces

In this section we collect some basic properties and general remarks concerning harmonic functions.

Let $\Gamma$ be a countable (infinite) group, $\mu$ a probability measure on $\Gamma$, and $X$ a compact topological space on which $\Gamma$ acts by homeomorphisms. It will always be assumed that $\mu$ is non-degenerate, i.e., the support of $\mu$ generates $\Gamma$ as a semigroup. In particular, $\mu(\gamma) < 1$ for all $\gamma \in \Gamma$. A continuous function $f : X \to \mathbb{R}$ is said to be $\mu$-harmonic if, for all $x \in X$,

$$f(x) = \sum_{\gamma \in \Gamma} f(\gamma(x))\mu(\gamma).$$

Let $H(X, \Gamma, \mu)$ denote the set of $\mu$-harmonic functions on $X$ and $C(\Gamma)$ the subset consisting of continuous $\Gamma$-invariant functions. If $H(X, \Gamma, \mu) = C(\Gamma)$, we say that $(X, \Gamma, \mu)$ satisfies the Liouville property.

We note that if $f \in H(X, \Gamma, \mu)$ and $x \in X$, then $\tilde{f}_x(\gamma) := f(\gamma^{-1}(x))$ is a harmonic function on $\Gamma$ in the following sense:

$$\tilde{f}_x(\gamma) = \sum_{\eta \in \Gamma} \tilde{f}_x(\eta\gamma)\mu(\eta^{-1}).$$

Under the conventions of [Fu], $\tilde{f}_x$ is harmonic relative to the right $\hat{\mu}$-random walk on $\Gamma$, where $\hat{\mu}(\gamma) = \mu(\gamma^{-1})$. Thus, if $(\Gamma, \hat{\mu})$ itself has the Liouville property, i.e., if bounded harmonic functions on $\Gamma$ are constant, then necessarily $(X, \Gamma, \mu)$ has the Liouville property for all $X$. Therefore, the present subject is uninteresting if the Poisson boundary of $(\Gamma, \hat{\mu})$ reduces to a single point. It is known, for example, that if $\Gamma$ is amenable, then there exists a symmetric ($\mu = \hat{\mu}$) non-degenerate $\mu$ for which the Poisson boundary is trivial.

A good example of group to keep in mind, on the other hand, is the fundamental group of a closed surface of genus at least 2, represented as isometries of the Poincaré disc. It has a nontrivial boundary for many interesting probability measures. For example, a result due to Furstenberg implies that on this group there exists a probability measure $\mu$ relative to which the boundary of $\Gamma$ coincides with the geometric boundary, $S^1$, of the disc [Fur].
Let $\Delta = P - I$ be the Laplace operator on the set $C(X)$ of real valued continuous functions on $X$, where $(P f)(x) := \sum \mu(\gamma) f(\gamma(x))$ is the averaging operator and $I$ is the identity. Clearly, $f$ is harmonic iff $\Delta f = 0$. More generally, $f$ satisfying $\Delta f \geq 0$ (resp., $\Delta f \leq 0$) is said to be superharmonic (resp., subharmonic).

**Proposition 2.1 (Maximum principle)** If a superharmonic $f \in C(X)$ is bounded above by $C$ and $f(x) = C$ for an $x \in X$, then $f$ is constant on the $\Gamma$-orbit of $x$.

**Proof.** This trivially follows from $\mu$ being non-degenerate and the remark

$$0 \leq \sum_{\gamma \in \Gamma} (f(\gamma(x)) - f(x)) \mu(\gamma) \leq \sum_{\gamma \in \Gamma} (f(\gamma(x)) - C) \mu(\gamma) \leq 0,$$

so $f(\gamma(x)) = C$ for all $\gamma$ in the support of $\mu$. \hfill $\square$

**Corollary 2.2** If the closure of every $\Gamma$-orbit contains a unique minimal set, then the Liouville property holds.

**Proof.** Let $X_0$ be the closure of an orbit. The maximum and minimum values in $X_0$ of a harmonic function are attained on orbits where the function is constant. Since $X_0$ has a single minimal set, these two values must agree. \hfill $\square$

It is useful to introduce a (directed) metric $d_{x_0}$ on the orbit $\Gamma x_0$ as follows. Given $x, y$ in $\Gamma x_0$, we specify a path joining $x$ to $y$ by a sequence $\alpha = (\gamma_0, \gamma_1, \ldots, \gamma_{n-1})$, $\mu(\gamma_i) > 0$, such that $y_0 = x$, $y_n = y$, and $y_{i+1} = \gamma_i y_i$ for $i = 0, \ldots, n - 1$. The length of $\alpha$ is defined as

$$L(\alpha) = \sum_{i=0}^{n-1} - \ln \mu(\gamma_i) \in (0, \infty].$$

We allow $x$ to be joined to itself by the trivial path, defined by $n = 0$, $\alpha = \emptyset$ and $L(\alpha) = 0$. Now let $d_{x_0}(x, y) = \inf L(\alpha)$, where the infimum is taken over all $n$ and all $(\gamma_0, \ldots, \gamma_{n-1})$ describing a path from $x$ to $y$.

**Proposition 2.3 (Harnack inequality)** Let $f$ be a continuous, positive, subharmonic function on $X$ and $x_0 \in X$ arbitrary. Then for all $x, y \in \Gamma x_0$,

$$\frac{f(y)}{f(x)} \leq e^{d_{x_0}(x, y)}.$$

**Proof.** As $f$ is subharmonic, $f(x) \geq \sum_{\gamma \in \Gamma} \mu(\gamma) f(\gamma x) \geq \mu(\eta) f(\eta x)$ for all $\eta \in \Gamma$, so $f(\eta x)/f(x) \leq e^{-\ln \mu(\eta)}$. Multiplying such inequalities along any path $\alpha$ from $x$ to $y$ yields the claim. \hfill $\square$

Given $(X, \Gamma, \mu)$ as above, a measure $\nu$ on $X$ is called harmonic if $\nu$ is $\Gamma$-quasi-invariant and $\mu * \nu = \nu$, where the convolution $\mu * \nu$ is the image of the product measure $\mu \otimes \nu$ under the action map $\Gamma \times X \to X$. It is easily shown that $\mu * \nu = \nu P$, where the averaging operator $P$ acts on $\nu$ by duality as

$$\nu P = \sum_{\gamma \in \Gamma} \mu(\gamma) \gamma * \nu.$$ 

The following useful remark is due, in the context of foliated Brownian motion, to Garnett [Gar]. The proof, which is sketched below for completeness, is adapted from Paulin [Pa].
Proposition 2.4 Let $\nu$ be a harmonic probability measure on $(X, \Gamma, \mu)$. Then every $h$ in $H(X, \Gamma, \mu)$ is $\Gamma$-invariant on the support of $\nu$.

Proof. It may be assumed that $h$ is non-negative. Let $c$ be a non-negative constant and define $h_c(x)$ as the minimum of $\{h(x), c\}$. Note that $Ph_c \leq h_c$. Since $\nu$ is harmonic, $\int h_c d\nu = \int h_c d(\mu \ast \nu) = \int Ph_c d\nu$, therefore $\int (Ph_c - h_c) d\nu = 0$. As the functions involved are continuous, it follows that $Ph_c = h_c$ on the support of $\nu$. Applying the maximum principle to $h_c|_{\text{supp}(\nu)}$, we conclude that $h^{-1}([c, \infty)) \cap \text{supp}(\nu)$ is $\Gamma$-invariant for all $c$. This implies that $h$ is $\Gamma$-invariant on $\text{supp}(\nu)$. \qed

Let $C$ denote the largest $\Gamma$-invariant subset of $X$ such that $h|_C$ is $\Gamma$-invariant for all $h \in H(X, \Gamma, \mu)$. Clearly, $C$ is closed. It is also non-empty since, by the maximum principle, it contains every minimal set in $X$. Thus if $M$ denotes the closure of the union of all the minimal sets of $X$ relative to the $\Gamma$-action, then $M \subseteq C$. Due to Proposition 2.4, the set $S$, defined as the closure of the union of supports of all harmonic measures, is also contained in $C$. In fact, it is easily seen that $M \subseteq S \subseteq C$. It will be explained shortly that the random walk associated to $(X, \Gamma, \mu)$ (defined next) converges to $C$ with probability 1.

3 Random walk on $(X, \Gamma, \mu)$

Let $(X, \Gamma, \mu)$ be, as above, a compact $\Gamma$-space where $\Gamma$ is a countable group and $\mu$ is a probability measure on $\Gamma$. The measure induces a Markov transition kernel on $X$ by setting $p(x, y) = 0$ if $y \notin \Gamma x$ and

$$p(x, y) = \sum_{\gamma \in \Gamma} \mu(\gamma)$$

where the sum is over all $\gamma \in \Gamma$ such that $\gamma x = y$. Let $X_n$ denote the random walk on $X$ with transition probabilities $p$ and initial state $x_0$. For the set $C$, defined at the end of the previous section, we have the following.

Proposition 3.1 The random walk $X_n$ on $X \setminus C$ is transient.

The proposition means that for all $x_0 \in X \setminus C$ and every open $U$ containing $C$, there exists a random integer $N$, almost surely finite, such that $X_n \in U$ for all $n \geq N$. Prior to proving this fact we need to review some information about the Poisson boundary of $\Gamma$. Our main source is Furman [Fu].

The boundary of $\Gamma$ can be described as a compact Hausdorff $\Gamma$-space $B$ endowed with a probability measure of full support, $\nu$, satisfying the following properties:

1. $\nu$ is harmonic relative to $(\Gamma, \hat{\mu})$;

2. For almost every sample path $\bar{\gamma} = (\gamma_1, \gamma_2, \ldots)$ of a (right) $\hat{\mu}$-random walk on $\Gamma$, a limit probability $\nu_\infty = \lim_{n \to \infty} (\gamma_1 \ldots \gamma_n) \ast \nu$ exists and is equal to a Dirac measure $\delta_{\eta_\infty}$ supported on a random point, $\eta_\infty$, of $B$;

3. Let $H^\infty(\Gamma, \hat{\mu})$ denote the space of bounded harmonic functions on $\Gamma$ relative to $\hat{\mu}$. Then the map $F : L^\infty(B, \nu) \to H^\infty(\Gamma, \hat{\mu})$ given by $F(\phi)(\gamma) = \int_B \phi d\gamma \ast \nu$ is an isometric bijection such that, for $h = F(\phi)$,

$$\phi(\eta_\infty) = \lim_{n \to \infty} h(\gamma_1 \ldots \gamma_n).$$

It is observed in [Fu], Remark 2.16 (i), that if $\Gamma$ is a discrete group then, for the topological description of $B$ given there (denoted $\overline{B}$ in Furman’s article), $L^\infty(B, \nu)$ coincides with the continuous functions on $B$. In this case, $B$ is non-metrizable if $\Gamma$ has non-trivial Poisson boundary. However, measure theoretically the Poisson boundary is always a Lebesgue space.
Lemma 3.2 Let \( \eta_1 = \gamma_1 \ldots \gamma_n \) be a right \( \tilde{\mu} \)-random walk starting at the unit element \( e \in \Gamma \). Let \( h \in H^\infty(\Gamma, \tilde{\mu}) \) and \( C \) a positive constant. Then, for a.e. sample path \( \eta_n \) and every sequence \( \xi_n \in \Gamma \) such that \( d_e(\eta_n, \xi_n) < C \), the sequences \( h(\eta_n) \) and \( h(\xi_n) \) converge to the same value. (Here \( d_e \) is the directed metric used in Proposition 2.3.)

**Proof.** Define \( \nu_\gamma = \gamma_* \nu \). From the above properties of the Poisson boundary and Proposition 2.3, it follows that

\[
\max \left\{ \frac{d\nu_{\eta_n}}{d\nu_{\xi_n}}, \frac{d\nu_{\xi_n}}{d\nu_{\eta_n}} \right\} \leq e^C
\]

for all \( n \). But since \( \nu_{\eta_n} \) converges to the Dirac measure \( \delta_{\eta_\infty} \), the sequence \( \nu_{\xi_n} \) must converge to the same measure. \( \square \)

We now return to Proposition 3.1. Let \( h \in H(X, \Gamma, \mu) \). As pointed out above, for each \( x \in X \setminus C \) we obtain a harmonic function \( \tilde{h}_x \in H^\infty(\Gamma, \tilde{\mu}) \). By Lemma 3.2, for each \( C > 0 \), for a.e. sample path \( (x_0, x_1, \ldots) \) of the random walk on \( X \), and for every sequence \( (y_0, y_1, \ldots) \) such that \( d_e(x_i, y_i) < C \) for all \( i \), both sequences \( \tilde{h}_x(x_i) \) and \( \tilde{h}_x(y_i) \) converge and have the same limit. Therefore, as \( C \) is arbitrary and \( h \) is continuous, \( h \) must be constant on the orbit of every limit point of \( x_i \). But this is to say that all limit points must lie in \( C \). This concludes the proof of Proposition 3.1.

One further simple remark, concerning induced random walks for subgroups, will be needed. Let \( \Gamma_0 \) be a finite index subgroup of \( \Gamma \). Let \( N^\gamma \) denote the first time, \( n \geq 1 \), at which random walk on \( (\Gamma, \tilde{\mu}) \) starting at \( \gamma \) reaches \( \Gamma_0 \). Then \( N^\gamma \) is a Markov time, almost surely finite. Now define a probability measure on \( \Gamma_0 \) by \( \tilde{\mu}_0(\eta) = \text{Prob}(\gamma_{N^\gamma} = \eta) \), that is, \( \tilde{\mu}_0(\eta) \) is the probability that random walk, \( (\gamma_0, \gamma_1, \ldots) \), on \( \Gamma \), starting at the unit element, will first return to \( \Gamma_0 \) at \( \eta \). General properties of Markov times and martingales imply that the restriction to \( \Gamma_0 \) of a function in \( H^\infty(\Gamma, \tilde{\mu}) \) is harmonic relative to \( \tilde{\mu}_0 \).

Lemma 3.3 Let \( \Gamma_0 \) have finite index in \( \Gamma \) and \( \mu_0 \) the image of the induced measure \( \tilde{\mu}_0 \) under group inverse. Then \( H(X, \Gamma, \mu) \subseteq H(X, \Gamma_0, \mu_0) \). In particular, if \( (X, \Gamma_0, \mu_0) \) satisfies the Liouville property, then so does \( (X, \Gamma, \mu) \).

**Proof.** This is due to the above remarks about \( \mu_0 \) and the relationship between a harmonic functions \( h \) on \( (X, \Gamma, \mu) \) and \( \tilde{h}_x \) on \( (\Gamma, \tilde{\mu}) \) or \( (\Gamma_0, \tilde{\mu}_0) \). (See the beginning of section 2.) \( \square \)

4 Actions on the circle

We now consider actions of discrete (countable) groups on the circle. The following lemma is a well-known group-actions version of Poincaré’s classification of circle homeomorphisms.

Lemma 4.1 Let \( \Gamma \) be a countable group of homeomorphisms of \( S^1 \). Then one of the following holds:

1. The action is minimal;
2. The action is not minimal and there is a unique minimal set;
3. There is a finite orbit.

**Proof.** It suffices to prove the lemma for a subgroup of finite index. Thus we may assume that \( \Gamma \) is orientation preserving. Further assume that the action is not minimal and contains no finite orbit. Let \( Z \) be a minimal set and \( U \) an arbitrary connected component of \( S^1 \setminus Z \). This minimal set is unique if the orbit of every point in \( U \) can be shown to accumulate on \( Z \). Now, \( U \) and \( \gamma U \) either
are disjoint or agree for each $\gamma \in \Gamma$. If the latter, then the end points of $U$ are fixed by $\gamma$. As the subgroup of $\Gamma$ fixing the end points of $U$ has infinite index, there are infinitely many disjoint intervals of the form $\gamma U$, so we can choose a sequence $\gamma_n U$ decreasing to 0 in length. Therefore, the orbit of every point in $U$ must limit on $Z$. □

By Corollary 2.2, having a unique minimal set implies the Liouville property. Thus, in order to prove Theorem 1.1, it suffices to assume that the action on $S^1$ contains a finite orbit. By Lemma 3.3, it can be assumed that $\Gamma$ has a fixed point in $S^1$ and preserves orientation. This reduces the proof of Theorem 1.1 to showing the Liouville property for orientation preserving actions on the interval $[0, 1]$.

5 Actions on the interval

We now restrict attention to systems $([0, 1], \Gamma, \mu)$ for which the $\Gamma$-action is orientation preserving. Clearly, then, the only finite orbits are fixed points, and since the union of the fixed points is a closed subset of $[0, 1]$, we may further restrict attention to intervals without interior finite orbits.

Lemma 5.1 If $\Gamma$ acts by orientation preserving homeomorphisms of $[0, 1]$ without interior finite orbits, the orbit of every $x \in (0, 1)$ must limit on both 0 and 1.

Proof. The supremum and infimum of $\Gamma x$ are easily seen to be fixed points. □

Lemma 5.2 Suppose that the Liouville property does not hold for $([0, 1], \Gamma, \mu)$ and the $\Gamma$-action is orientation preserving without interior finite orbits. Then:

1. There is a unique continuous harmonic $f$ not $\Gamma$-invariant such that $f(0) = 0, f(1) = 1$;
2. $f(x)$ is the probability that random walk on $([0, 1], \Gamma, \mu)$ starting at $x$ converges to 1;
3. $f$ is not constant on any interior orbit;
4. $f$ is increasing.

Proof. Let $g$ be a continuous, harmonic, non-$\Gamma$-invariant function on $[0, 1]$. By Lemma 5.1 and the maximum principle, $g(0) \neq g(1)$, and $g(x)$ lies in the open interval with end points $g(0), g(1)$ for each $x \in (0, 1)$. Lemma 5.1 also implies that $g$ cannot be constant on any interior orbit, so the set $C$ of Proposition 3.1 coincides with $\{0, 1\}$. By composing $g$ with an appropriate affine function of $\mathbb{R}$, one obtains $f$ such that $f(i) = i$ for $i = 0, 1$ and $0 < f(x) < 1$ on interior points. Uniqueness is due to the maximum principle. By Proposition 3.1, the random walk $X_n^x$ on $[0, 1]$ starting at $x$ must converge to a random point $X_\infty \in \{0, 1\}$ with probability 1. As $f$ is harmonic, the expected value $E[f(X_\infty^n)]$ is equal to $f(x)$ and $\lim_{n \to \infty} f(X_n^x)$ exists almost surely. By continuity, $f(X_n^x)$ converges to either 0 or 1. Therefore,

$$f(x) = \lim_{n \to \infty} E[f(X_n^x)] = f(0) \text{Prob}(X_n^x \to 0) + f(1) \text{Prob}(X_n^x \to 1) = \text{Prob}(X_n^x \to 1).$$

Finally, given any $x_1 < x_2$ and a sample path $(\gamma_1, \gamma_2, \ldots)$ on $\Gamma$ for the right $\hat{\mu}$-random walk, the corresponding sample paths for the random walks on $[0, 1]$ satisfy $X_n^{x_1} < X_n^{x_2}$. Therefore, the probability that $X_n^{x_2}$ converges to 1 is at least as great as the probability that $X_n^{x_1}$ converges to 1. This shows that $f(x)$ is increasing. □
We can now show how the assumption of a non-$\Gamma$-invariant harmonic function on $[0, 1]$ leads to a contradiction. Let $f$ be as in Lemma 5.2 and define a probability measure $\nu$ on $[0, 1]$ by extending the definition 

$$\nu((a, b]) = f(b) - f(a)$$

to the Lebesgue measurable subsets of the interval. At this point we make the further assumption that the measure $\mu$ on $\Gamma$ is symmetric. Then,

$$\mu \ast \nu((a, b]) = \sum_{\gamma \in \Gamma} \mu(\gamma) \nu((\gamma^{-1}a, \gamma^{-1}b])$$

$$= \sum_{\gamma \in \Gamma} \mu(\gamma) (f(\gamma^{-1}b) - f(\gamma^{-1}a))$$

$$= \sum_{\gamma \in \Gamma} \mu(\gamma) f(\gamma b) - \sum_{\gamma \in \Gamma} \mu(\gamma) f(\gamma a)$$

$$= f(b) - f(a) = \nu((a, b]).$$

This remark, which was shown to us by B. Deroin, simplifies a similar but somewhat more involved argument of an earlier version of this paper. The same argument is used in the proof of Proposition 5.7 of [DKN]. From this we obtain the following lemma.

**Lemma 5.3** If $\mu$ is symmetric, $\nu$ is a harmonic probability measure.

We can now conclude the proof of Theorem 1.1. Since by Proposition 2.4 any function in $H(X, \Gamma, \mu)$ must be $\Gamma$-invariant on the support of a harmonic probability measure, and since the above $f$ is not constant on any interior orbit of the $\Gamma$-action on $[0, 1]$, we arrive at a contradiction. Therefore, a non-$\Gamma$-invariant, continuous harmonic function cannot exist.

### 6 Examples

We give now a class of examples of $\Gamma$-spaces to illustrate the way in which the Liouville property can fail to hold. We begin with a general remark that suggests a recipe for constructing examples. Let $\mu$ be a probability measure on the countable (discrete) group $\Gamma$ and let $\hat{\mu}$, as before, be the image of $\mu$ under group inverse. The unit ball in $H(\Gamma, \hat{\mu})$ (the latter equipped with the supremum norm) is a compact space, which we denote by $X_0$. Note that $\Gamma$ acts on $X_0$ by homeomorphisms under the definition $(\gamma, \phi) \mapsto \gamma \cdot \phi$, where $(\gamma \cdot \phi)(\eta) = \phi(\gamma^{-1}\eta)$.

The $\Gamma$-space $X_0$ has a tautological continuous $\mu$-harmonic function: $f(\phi) = \phi(e)$. This is, in fact, $\mu$-harmonic since

$$\sum_{\gamma \in \Gamma} f(\gamma \cdot \phi) \mu(\gamma) = \sum_{\gamma \in \Gamma} \phi(\gamma^{-1}) \mu(\gamma) = \sum_{\gamma \in \Gamma} \phi(\gamma) \hat{\mu}(\gamma) = \phi(e) = f(\phi).$$

Furthermore, since $f(\gamma \phi) = \phi(\gamma^{-1})$, $f$ is not constant on $\Gamma \cdot \phi$ if $\phi$ itself is not a constant function. This simple remark suggests the following approach to finding examples of non-Liouville actions on a $\Gamma$-space $S$. Suppose we can somehow construct a continuous, $\Gamma$-equivariant map $\Phi : S \to X_0$. We express equivariance by $\Phi(\gamma \cdot s) = \gamma \Phi(s)$ and write $\Phi(s) = \phi_s$. If $\Phi$ does not map $S$ entirely into the space of constant functions, then $f \circ \Phi$ defines a continuous, $\mu$-harmonic function on $S$ which is not $\Gamma$-invariant. Thus, what we have shown in Theorem 1.1 amounts to the following.

**Proposition 6.1** If $\mu$ is a symmetric probability measure on $\Gamma$, then every $\Gamma$-equivariant continuous $\Phi : S^1 \to X_0$ maps into the constant functions.
We now construct a continuous $\Gamma$-equivariant map $\Phi : S^2 \to X_0$ whose image is not contained in the space of constant functions. By the above remark, this yields a non-Liouville action on $S^2$. The example generalizes one given in [FZ2].

Let $\Gamma$ and $\hat{\mu}$ be such that the Poisson boundary of $(\Gamma, \hat{\mu})$ can be identified with the circle $S^1$. For example, as already noted, if $\Gamma$ is a uniform group of isometries of the Poincaré disc, it is possible to find a symmetric $\mu$ for which the Poisson boundary of the group coincides with the geometric boundary of the disc [Fur]. Let $\nu$ be a harmonic measure on $S^1$ as in the discussion immediately after Proposition 3.1. The following general fact was pointed out to us by Kaimanovich.

**Lemma 6.2** Let $\hat{\mu}$ be a non-degenerate probability measure on $\Gamma$ such that a Poisson boundary $(B, \nu)$ of $(\Gamma, \hat{\mu})$ is non-trivial. Then the $\hat{\mu}$-harmonic measure $\nu$ has no atoms.

**Proof.** Suppose for a contradiction that $\nu$ does have atoms, and let $b \in B$ be such that $\nu(b) \geq \nu(b')$ for all $b' \in B$. Since $\nu$ is $\hat{\mu}$-harmonic and $\hat{\mu}$ is non-degenerate, the maximum principle applied to $\gamma \mapsto (\gamma, \nu)(b)$ implies $\nu(b) = \nu(\gamma b)$ for all $\gamma \in \Gamma$. On the other hand, if $(\gamma_1, \gamma_2, \ldots)$ is a random walk on $(\Gamma, \hat{\mu})$, then $(\gamma_1 \ldots \gamma_n) \nu$ converges to the Dirac measure $\delta_{\gamma b}$, where $\gamma b$ is the random point on the boundary to which the random walk converges. Writing $g_n = \gamma_1 \ldots \gamma_n$, then $0 < \nu(b) = (g_n \nu)(b) \to \delta_{\gamma b}(b)$. This is only possible if sample paths converge to $b$ almost surely, contradicting the assumption that the Poisson boundary is non-trivial. □

We now construct a non-trivial $\Gamma$-equivariant continuous map $\Phi : S^2 \to X_0$. Let $x = (z, \theta)$ in the cylinder $S^1 \times [0, 2\pi]$ represent the interval $I_x \subset S^1$ with endpoints $z$ and $ze^{i\theta}$. Naturally, $\gamma I_x = I_{\gamma x}$. The circle boundaries $S^1 \times \{0\}$ and $S^1 \times \{2\pi\}$ are invariant under $\Gamma$ and correspond to intervals of length $0$ or $2\pi$. The quotient $(S^1 \times [0, 2\pi]) / \sim$ obtained by collapsing the boundary circles to points is homeomorphic to $S^2$ and the $\Gamma$-action on the cylinder defines an action on the quotient by homeomorphisms fixing two points, denoted $N = (S^1 \times \{2\pi\}) / \sim$ and $S = (S^1 \times \{0\}) / \sim$. This defines $S^2$ as a topological $\Gamma$-space.

Let $\nu$ be a harmonic probability measure on $S^1$, which is granted since $S^1$ is a Poisson boundary of $(\Gamma, \hat{\mu})$. Define $\Phi(x) = \phi_x$, where $\phi_x(\gamma) = \gamma \nu(I_x)$. Note that $\Phi$ is well-defined since $\nu(I_x) = 0$ on all intervals of length $0$, and $1$ on all intervals of length $2\pi$. A simple calculation shows that $\phi_x$ is $\hat{\mu}$-harmonic for each $x \in S^2$ and that $\Phi$ is $\Gamma$-equivariant. Furthermore, $\Phi$ is continuous due to the fact that $\nu$ has no atoms. Since the Poisson boundary is non-trivial, $\Phi$ does not map into the set of constant functions. This shows that $(S^2, \Gamma, \mu)$ is non-Liouville.

In this example, the only $\Gamma$-orbits in $S^2$ on which $\Phi(x)$ is a constant function are the fixed points $N, S$, so $C = \{N, S\}$, where $C$ is the set defined in Proposition 3.1. The same argument used in Lemma 5.2 shows that $g = f \circ \Phi$ can be composed with an affine transformation of $\mathbb{R}$ to insure $g(S) = 0$ and $g(N) = 1$, after which $g(x)$ is the probability that random walk on $(S^2, \Gamma, \mu)$ starting at $x \in S^2$ converges to $N$.

When $(S^1, \nu)$ coincides with the boundary of the hyperbolic disc, where $\Gamma$ is a uniform lattice in $PSL(2, \mathbb{R})$, Lemma 4.2 of [FZ2] can be used to show that the just constructed action on $S^2$ is ergodic relative to the smooth measure class on $S^2$. This dynamical property of the action contrasts with the simple behavior of the random walk, which converges to $N$ or $S$ with probability $1$.

It is interesting to regard the induced random walk on the space $X_0$ itself, for a general $(\Gamma, \hat{\mu})$. Let $I = [-1, 1] \subset X_0$ represent the subspace of constant functions in $X_0$. Then the random walk on $(X_0, \Gamma, \mu)$ converges towards $I$, i.e., the random walk on $X_0 \setminus I$ is transient. On the other hand, the $\Gamma$-action on $X_0$ can be dynamically very complicated. For example, it is shown in [FZ2] for subgroups of $PSL(2, \mathbb{R})$ that the $Z$-action on $X_0$ induced by a parabolic or hyperbolic element is a chaotic dynamical system.
References


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