Harmonic functions over group actions

Renato Feres and Emily Ronshausen

To Bob Zimmer, on his 60th birthday

Abstract

Let X be a compact space, Γ a countable group of homeomorphisms of X, and μ a probability measure on Γ. From this data one naturally defines a random walk on X and a related notion of continuous harmonic function. The question we wish to investigate is whether all continuous harmonic functions on X are Γ-invariant. If this is the case we say that the system satisfies the topological *Liouville property*. We show that if μ is a symmetric probability measure on an arbitrary Γ and $X = S^1$ or the interval $[0, 1]$, then the Liouville property always holds. In addition, when the Poisson boundary of (Γ, μ) can be identified with the circle, we give a general construction of a non-Liouville action of Γ on S^2, generalizing an example from [FZ2].

1 Introduction

A useful idea in the study of group actions without invariant measures is to introduce and study a random $\mathbb{Z}$-action associated to a choice of probability measure on the acting group. To fix notations, let Γ denote a group of transformations of a space X and μ a probability measure on Γ. Then μ defines a random walk on Γ, which induces a random $\mathbb{Z}$-action on X. See, for example, [Fu] where this point of view is well-represented. It is often assumed in this context that the Γ-space admits a μ-stationary measure (also called μ-harmonic measure—see definition below) of full support, endowing the random $\mathbb{Z}$-action with an invariant measure. This brings along the ergodic theory apparatus that relies on the existence of invariant measures.

Often, however, the Γ-space has large regions with possibly interesting dynamics that are not detected by any harmonic measure for a given μ. If this is the case, one can hope to gain some insight into the system represented by (X, Γ, μ) by investigating the space of μ-harmonic functions on X. As pointed out in Proposition 2.4 below, continuous μ-harmonic functions on compact Γ-spaces are Γ-invariant on the support of any μ-harmonic probability measure, so non-trivial (i.e., non-Γ-invariant) such functions may yield information about the regions that harmonic measures do not detect. This is the perspective motivating this paper. We have chosen here to consider compact Γ-spaces and continuous functions, although one may also try to pursue the subject on measurable Γ-spaces.

The most basic question to ask about (X, Γ, μ) in the present context is whether it satisfies a topological *Liouville property*, i.e., whether all continuous μ-harmonic functions are Γ-invariant. The main result of the paper is that the Liouville property holds for actions of countable groups on one-dimensional spaces when μ is a symmetric probability measure.

Theorem 1.1 *Let Γ be a countable group acting on X by homeomorphisms, where X is either the circle S^1 or the interval $[0, 1]$. Let μ be a symmetric probability measure on Γ. Then the Liouville property holds for (X, Γ, μ).*

We give at the end of the paper an example of Γ-space for which the Liouville property does not hold. More specifically, we show that if the Poisson boundary of (Γ, μ) can be identified with the circle S^1, then it is possible to construct a Γ-action on the two-sphere so that (S^2, Γ, μ) admits continuous μ-harmonic functions that are non-constant on all orbits in the complement of a pair of fixed points. The action can be chosen to be ergodic with respect to the smooth measure class on S^2, whereas the random walk is transient in the complement of the two fixed points. A general recipe for constructing non-Liouville actions is also proposed.

The proof of Theorem 1.1 parallels the proof of the main theorem of [FFP]. That paper is about the Liouville property for codimension-one foliated Brownian motion and harmonic functions for the Laplace-Beltrami operator associated to a Riemannian metric on leaves, rather than discrete group actions. Although the two situations cannot be directly compared, our assumptions on Γ and μ are, in a sense, much more general than related assumptions on that paper. The example of a non-Liouville action on S^2 generalizes a similar construction given in [FZ2] in the context of foliations with a Riemannian metric on leaves. See, also, [FZ1] for these issues in the setting of holomorphic functions on foliations. The paper [DK] has some deep results about harmonic functions on foliated spaces in the spirit of the present discussion.

This paper was partly written during a visit by the first author to the Paris-Sud 11 University, at Orsay. He wishes to thank the group of Topology and Dynamics for their hospitality and the university for its generous support. He also wishes to thank Vadim Kaimanovich and Bertrand Deroin for helpful and illuminating discussions. In addition, we wish to express our thanks to the anonymous referee for several helpful comments.

2 Harmonic functions on Γ-spaces

In this section we collect some basic properties and general remarks concerning harmonic functions.

Let Γ be a countable (infinite) group, μ a probability measure on Γ, and X a compact topological space on which Γ acts by homeomorphisms. It will always be assumed that μ is *non-degenerate*, i.e., the support of μ generates Γ as a semigroup. In particular, $\mu(\gamma) < 1$ for all $\gamma \in \Gamma$. A continuous function $f : X \to \mathbb{R}$ is said to be *μ-harmonic* if, for all $x \in X$,

$$f(x) = \sum_{\gamma \in \Gamma} f(\gamma(x))\mu(\gamma).$$

Let $H(X, \Gamma, \mu)$ denote the set of μ-harmonic functions on X and $C(X)^\Gamma$ the subset consisting of continuous Γ-invariant functions. If $H(X, \Gamma, \mu) = C(X)^\Gamma$, we say that (X, Γ, μ) satisfies the *Liouville property*.

We note that if $f \in H(X, \Gamma, \mu)$ and $x \in X$, then $\tilde{f}_x(\gamma) := f(\gamma^{-1}(x))$ is a harmonic function on Γ in the following sense:

$$\tilde{f}_x(\gamma) = \sum_{\eta \in \Gamma} \tilde{f}_x(\gamma\eta)\mu(\eta^{-1}).$$

Under the conventions of [Fu], $\tilde{f}_x$ is harmonic relative to the right $\hat{\mu}$-random walk on Γ, where $\hat{\mu}(\gamma) = \mu(\gamma^{-1})$. Thus, if $(\Gamma, \hat{\mu})$ itself has the Liouville property, i.e., if bounded harmonic functions on Γ are constant, then necessarily (X, Γ, μ) has the Liouville property for all X. Therefore, the present subject is uninteresting if the Poisson boundary of $(\Gamma, \hat{\mu})$ reduces to a single point. It is known, for example, that if Γ is amenable, then there exists a symmetric ($\mu = \hat{\mu}$) non-degenerate μ for which the Poisson boundary is trivial.

A good example of group to keep in mind, on the other hand, is the fundamental group of a closed surface of genus at least 2, represented as isometries of the Poincaré disc. It has a nontrivial boundary for many interesting probability measures. For example, a result due to Furstenberg implies that on this group there exists a probability measure μ relative to which the boundary of Γ coincides with the geometric boundary, S^1, of the disc [Fur].

Let $\Delta = P - I$ be the *Laplace operator* on the set $C(X)$ of real valued continuous functions on X, where $(Pf)(x) := \sum \mu(\gamma) f(\gamma(x))$ is the averaging operator and I is the identity. Clearly, f is harmonic iff $\Delta f = 0$. More generally, f satisfying $\Delta f \geq 0$ (resp., $\Delta f \leq 0$) is said to be *superharmonic* (resp., *subharmonic*).

Proposition 2.1 (Maximum principle) *If a superharmonic $f \in C(X)$ is bounded above by C and $f(x) = C$ for an $x \in X$, then f is constant on the Γ-orbit of x.*

Proof. This trivially follows from μ being non-degenerate and the remark

$$0 \leq \sum_{\gamma \in \Gamma} (f(\gamma(x)) - f(x))\mu(\gamma) \leq \sum_{\gamma \in \Gamma} (f(\gamma(x)) - C)\mu(\gamma) \leq 0,$$

so $f(\gamma(x)) = C$ for all γ in the support of μ. □

Corollary 2.2 *If the closure of every Γ-orbit contains a unique minimal set, then the Liouville property holds.*

Proof. Let X_0 be the closure of an orbit. The maximum and minimum values in X_0 of a harmonic function are attained on orbits where the function is constant. Since X_0 has a single minimal set, these two values must agree. □

It is useful to introduce a (directed) metric d_{x_0} on the orbit Γx_0 as follows. Given x, y in Γx_0, we specify a *path* joining x to y by a sequence $\alpha = (\gamma_0, \gamma_1, \ldots, \gamma_{n-1})$, $\mu(\gamma_i) > 0$, such that $y_0 = x$, $y_n = y$, and $y_{i+1} = \gamma_i y_i$ for $i = 0, \ldots, n-1$. The *length* of α is defined as

$$L(\alpha) = \sum_{i=0}^{n-1} - \ln \mu(\gamma_i) \in (0, \infty].$$

We allow x to be joined to itself by the trivial path, defined by $n = 0$, $\alpha = \emptyset$ and $L(\alpha) = 0$. Now let $d_{x_0}(x, y) = \inf L(\alpha)$, where the infimum is taken over all n and all $(\gamma_0, \ldots, \gamma_{n-1})$ describing a path from x to y.

Proposition 2.3 (Harnack inequality) *Let f be a continuous, positive, subharmonic function on X and $x_0 \in X$ arbitrary. Then for all $x, y \in \Gamma x_0$,*

$$\frac{f(y)}{f(x)} \leq e^{d_{x_0}(x,y)}.$$

Proof. As f is subharmonic, $f(x) \geq \sum_{\gamma \in \Gamma} \mu(\gamma) f(\gamma x) \geq \mu(\eta) f(\eta x)$ for all $\eta \in \Gamma$, so $f(\eta x)/f(x) \leq e^{-\ln \mu(\eta)}$. Multiplying such inequalities along any path α from x to y yields the claim. □

Given (X, Γ, μ) as above, a measure ν on X is called *harmonic* if ν is Γ-quasi-invariant and $\mu * \nu = \nu$, where the convolution $\mu * \nu$ is the image of the product measure $\mu \otimes \nu$ under the action map $\Gamma \times X \to X$. It is easily shown that $\mu * \nu = \nu P$, where the averaging operator P acts on ν by duality as

$$\nu P = \sum_{\gamma \in \Gamma} \mu(\gamma) \gamma_* \nu.$$

The following useful remark is due, in the context of foliated Brownian motion, to Garnett [Gar]. The proof, which is sketched below for completeness, is adapted from Paulin [Pa].

Proposition 2.4 *Let ν be a harmonic probability measure on (X, Γ, μ). Then every h in $H(X, \Gamma, \mu)$ is Γ-invariant on the support of ν.*

Proof. It may be assumed that h is non-negative. Let c be a non-negative constant and define $h_c(x)$ as the minimum of $\{h(x), c\}$. Note that $Ph_c \leq h_c$. Since ν is harmonic, $\int h_c d\nu = \int h_c d(\mu * \nu) = \int Ph_c d\nu$, therefore $\int (Ph_c - h_c) d\nu = 0$. As the functions involved are continuous, it follows that $Ph_c = h_c$ on the support of ν. Applying the maximum principle to $h_c|_{\text{supp}(\nu)}$, we conclude that $h^{-1}([c, \infty)) \cap \text{supp}(\nu)$ is Γ-invariant for all c. This implies that h is Γ-invariant on $\text{supp}(\nu)$. □

Let $\mathcal{C}$ denote the largest Γ-invariant subset of X such that $h|_{\mathcal{C}}$ is Γ-invariant for all $h \in H(X, \Gamma, \mu)$. Clearly, $\mathcal{C}$ is closed. It is also non-empty since, by the maximum principle, it contains every minimal set in X. Thus if $\mathcal{M}$ denotes the closure of the union of all the minimal sets of X relative to the Γ-action, then $\mathcal{M} \subseteq \mathcal{C}$. Due to Proposition 2.4, the set $\mathcal{S}$, defined as the closure of the union of supports of all harmonic measures, is also contained in $\mathcal{C}$. In fact, it is easily seen that $\mathcal{M} \subseteq \mathcal{S} \subseteq \mathcal{C}$. It will be explained shortly that the random walk associated to (X, Γ, μ) (defined next) converges to $\mathcal{C}$ with probability 1.

3 Random walk on (X, Γ, μ)

Let (X, Γ, μ) be, as above, a compact Γ-space where Γ is a countable group and μ is a probability measure on Γ. The measure induces a Markov transition kernel on X by setting $p(x, y) = 0$ if $y \notin \Gamma x$ and

$$p(x, y) = \sum_{y = \gamma x} \mu(\gamma)$$

where the sum is over all $\gamma \in \Gamma$ such that $\gamma x = y$. Let X_n denote the random walk on X with transition probabilities p and initial state x_0. For the set $\mathcal{C}$, defined at the end of the previous section, we have the following.

Proposition 3.1 *The random walk X_n on $X \setminus \mathcal{C}$ is transient.*

The proposition means that for all $x_0 \in X \setminus \mathcal{C}$ and every open U containing $\mathcal{C}$, there exists a random integer N, almost surely finite, such that $X_n \in U$ for all $n \geq N$. Prior to proving this fact we need to review some information about the Poisson boundary of Γ. Our main source is Furman [Fu].

The boundary of Γ can be described as a compact Hausdorff Γ-space B endowed with a probability measure of full support, ν, satisfying the following properties:

1. ν is harmonic relative to $(\Gamma, \hat{\mu})$;

2. For almost every sample path $\bar{\gamma} = (\gamma_1, \gamma_2, \dots)$ of a (right) $\hat{\mu}$-random walk on Γ, a limit probability $\nu_{\bar{\gamma}} = \lim_{n \to \infty} (\gamma_1 \dots \gamma_n)_* \nu$ exists and is equal to a Dirac measure δ_{γ_∞} supported on a random point, η_∞, of B;

3. Let $H^\infty(\Gamma, \hat{\mu})$ denote the space of bounded harmonic functions on Γ relative to $\hat{\mu}$. Then the map $F : L^\infty(B, \nu) \to H^\infty(\Gamma, \hat{\mu})$ given by $F(\phi)(\gamma) = \int_B \phi \, d\gamma_* \nu$ is an isometric bijection such that, for $h = F(\phi)$,
$$\phi(\eta_\infty) = \lim_{n \to \infty} h(\gamma_1 \dots \gamma_n).$$

It is observed in [Fu], Remark 2.16 (i), that if Γ is a discrete group then, for the topological description of B given there (denoted $\overline{B}$ in Furman's article), $L^\infty(B, \nu)$ coincides with the continuous functions on B. In this case, B is non-metrizable if Γ has non-trivial Poisson boundary. However, measure theoretically the Poisson boundary is always a Lebesgue space.

Lemma 3.2 *Let $\eta_n = \gamma_1 \dots \gamma_n$ be a right $\hat{\mu}$-random walk starting at the unit element $e \in \Gamma$. Let $h \in H^\infty(\Gamma, \hat{\mu})$ and C a positive constant. Then, for a.e. sample path η_n and every sequence $\xi_n \in \Gamma$ such that $d_e(\eta_n, \xi_n) < C$, the sequences $h(\eta_n)$ and $h(\xi_n)$ converge to the same value. (Here d_e is the directed metric used in Proposition 2.3.)*

Proof. Define $\nu_\gamma = \gamma_* \nu$. From the above properties of the Poisson boundary and Proposition 2.3, it follows that

$$\max \left\{ \frac{d\nu_{\eta_n}}{d\nu_{\xi_n}}, \frac{d\nu_{\xi_n}}{d\nu_{\eta_n}} \right\} \leq e^C$$

for all n. But since ν_{η_n} converges to the Dirac measure δ_{η_∞}, the sequence ν_{ξ_n} must converge to the same measure. ☐

We now return to Proposition 3.1. Let $h \in H(X, \Gamma, \mu)$. As pointed out above, for each $x \in X \setminus \mathcal{C}$ we obtain a harmonic function $\tilde{h}_x \in H^\infty(\Gamma, \hat{\mu})$. By Lemma 3.2, for each $C > 0$, for a.e. sample path $(x_0, x_1, \dots)$ of the random walk on X, and for every sequence $(y_0, y_1, \dots)$ such that $d_x(x_i, y_i) < C$ for all i, both sequences $\tilde{h}_x(x_i)$ and $\tilde{h}_x(y_i)$ converge and have the same limit. Therefore, as C is arbitrary and h is continuous, h must be constant on the orbit of every limit point of x_i. But this is to say that all limit points must lie in $\mathcal{C}$. This concludes the proof of Proposition 3.1.

One further simple remark, concerning induced random walks for subgroups, will be needed. Let Γ_0 be a finite index subgroup of Γ. Let N^γ denote the first time, $n \geq 1$, at which random walk on $(\Gamma, \hat{\mu})$ starting at γ reaches Γ_0. Then N^γ is a Markov time, almost surely finite. Now define a probability measure on Γ_0 by $\hat{\mu}_0(\eta) = \text{Prob}(\gamma_{N^e} = \eta)$, that is, $\hat{\mu}_0(\eta)$ is the probability that random walk, $(\gamma_0, \gamma_1, \dots)$, on Γ, starting at the unit element, will first return to Γ_0 at η. General properties of Markov times and martingales imply that the restriction to Γ_0 of a function in $H^\infty(\Gamma, \hat{\mu})$ is harmonic relative to $\hat{\mu}_0$.

Lemma 3.3 *Let Γ_0 have finite index in Γ and μ_0 the image of the induced measure $\hat{\mu}_0$ under group inverse. Then $H(X, \Gamma, \mu) \subseteq H(X, \Gamma_0, \mu_0)$. In particular, if (X, Γ_0, μ_0) satisfies the Liouville property, then so does (X, Γ, μ).*

Proof. This is due to the above remarks about μ_0 and the relationship between a harmonic functions h on (X, Γ, μ) and $\tilde{h}_x$ on $(\Gamma, \hat{\mu})$ or $(\Gamma_0, \hat{\mu}_0)$. (See the beginning of section 2.) ☐

4 Actions on the circle

We now consider actions of discrete (countable) groups on the circle. The following lemma is a well-known group-actions version of Poincaré's classification of circle homeomorphisms.

Lemma 4.1 *Let Γ be a countable group of homeomorphisms of S^1. Then one of the following holds:*

1. *The action is minimal;*
2. *The action is not minimal and there is a unique minimal set;*
3. *There is a finite orbit.*

Proof. It suffices to prove the lemma for a subgroup of finite index. Thus we may assume that Γ is orientation preserving. Further assume that the action is not minimal and contains no finite orbit. Let Z be a minimal set and U an arbitrary connected component of $S^1 \setminus Z$. This minimal set is unique if the orbit of every point in U can be shown to accumulate on Z. Now, U and γU either are disjoint or agree for each $\gamma \in \Gamma$. If the latter, then the end points of U are fixed by γ. As the subgroup of Γ fixing the end points of U has infinite index, there are infinitely many disjoint intervals of the form γU, so we can choose a sequence $\gamma_n U$ decreasing to 0 in length. Therefore, the orbit of every point in U must limit on Z. □

By Corolary 2.2, having a unique minimal set implies the Liouville property. Thus, in order to prove Theorem 1.1, it suffices to assume that the action on S^1 contains a finite orbit. By Lemma 3.3, it can be assumed that Γ has a fixed point in S^1 and preserves orientation. This reduces the proof of Theorem 1.1 to showing the Liouville property for orientation preserving actions on the interval $[0, 1]$.

5 Actions on the interval

We now restrict attention to systems $([0, 1], \Gamma, \mu)$ for which the Γ-action is orientation preserving. Clearly, then, the only finite orbits are fixed points, and since the union of the fixed points is a closed subset of $[0, 1]$, we may further restrict attention to intervals without interior finite orbits.

Lemma 5.1 *If Γ acts by orientation preserving homeomorphisms of $[0, 1]$ without interior finite orbits, the orbit of every $x \in (0, 1)$ must limit on both 0 and 1.*

Proof. The supremum and infimum of Γx are easily seen to be fixed points. □

Lemma 5.2 *Suppose that the Liouville property does not hold for $([0, 1], \Gamma, \mu)$ and the Γ-action is orientation preserving without interior finite orbits. Then:*

1. *There is a unique continuous harmonic f not Γ-invariant such that $f(0) = 0$, $f(1) = 1$;*
2. *$f(x)$ is the probability that random walk on $([0, 1], \Gamma, \mu)$ starting at x converges to 1;*
3. *f is not constant on any interior orbit;*
4. *f is increasing.*

Proof. Let g be a continuous, harmonic, non-Γ-invariant function on $[0, 1]$. By Lemma 5.1 and the maximum principle, $g(0) \neq g(1)$, and $g(x)$ lies in the open interval with end points $g(0), g(1)$ for each $x \in (0, 1)$. Lemma 5.1 also implies that g cannot be constant on any interior orbit, so the set $\mathcal{C}$ of Proposition 3.1 coincides with $\{0, 1\}$. By composing g with an appropriate affine function of $\mathbb{R}$, one obtains f such that $f(i) = i$ for $i = 0, 1$ and $0 < f(x) < 1$ on interior points. Uniqueness is due to the maximum principle. By Proposition 3.1, the random walk X_n^x on $[0, 1]$ starting at x must converge to a random point $X_\infty \in \{0, 1\}$ with probability 1. As f is harmonic, the expected value $E[f(X_n^x)]$ is equal to $f(x)$ and $\lim_{n\to\infty} f(X_n)$ exists almost surely. By continuity, $f(X_n^x)$ converges to either 0 or 1. Therefore,

$$f(x) = \lim_{n\to\infty} E[f(X_n^x)] = f(0)\text{Prob}(X_n^x \to 0) + f(1)\text{Prob}(X_n^x \to 1) = \text{Prob}(X_n^x \to 1).$$

Finally, given any $x_1 < x_2$ and a sample path $(\gamma_1, \gamma_2, \dots)$ on Γ for the right $\hat{\mu}$-random walk, the corresponding sample paths for the random walks on $[0, 1]$ satisfy $X_n^{x_1} < X_n^{x_2}$. Therefore, the probability that $X_n^{x_2}$ converges to 1 is at least as great as the probability that $X_n^{x_1}$ converges to 1. This shows that $f(x)$ is increasing. □

We can now show how the assumption of a non-Γ-invariant harmonic function on $[0,1]$ leads to a contradiction. Let f be as in Lemma 5.2 and define a probability measure ν on $[0,1]$ by extending the definition

$$\nu((a,b]) = f(b) - f(a)$$

to the Lebesgue measurable subsets of of the interval. At this point we make the further assumption that the measure μ on Γ is symmetric. Then,

$$\begin{aligned}
\mu * \nu((a,b]) &= \sum_{\gamma\in\Gamma} \mu(\gamma)\nu\left((\gamma^{-1}a, \gamma^{-1}b]\right) \\
&= \sum_{\gamma\in\Gamma} \mu(\gamma)\left(f(\gamma^{-1}b) - f(\gamma^{-1}a)\right) \\
&= \sum_{\gamma\in\Gamma} \mu(\gamma) f(\gamma b) - \sum_{\gamma\in\Gamma} \mu(\gamma) f(\gamma a) \\
&= f(b) - f(a) \\
&= \nu((a,b]).
\end{aligned}$$

This remark, which was shown to us by B. Deroin, simplifies a similar but somewhat more involved argument of an earlier version of this paper. The same argument is used in the proof of Proposition 5.7 of [DKN]. From this we obtain the following lemma.

Lemma 5.3 *If μ is symmetric, ν is a harmonic probability measure.*

We can now conclude the proof of Theorem 1.1. Since by Proposition 2.4 any function in $H(X,\Gamma,\mu)$ must be Γ-invariant on the support of a harmonic probability measure, and since the above f is not constant on any interior orbit of the Γ-action on $[0,1]$, we arrive at a contradiction. Therefore, a non-Γ-invariant, continuous harmonic function cannot exist.

6 Examples

We give now a class of examples of Γ-spaces to illustrate the way in which the Liouville property can fail to hold. We begin with a general remark that suggests a recipe for constructing examples. Let μ be a probability measure on the countable (discrete) group Γ and let $\hat{\mu}$, as before, be the image of μ under group inverse. The unit ball in $H^\infty(\Gamma,\hat{\mu})$ (the latter equipped with the supremum norm) is a compact space, which we denote by X_0. Note that Γ acts on X_0 by homeomorphisms under the definition $(\gamma,\phi)\mapsto \gamma\cdot\phi$, where $(\gamma\cdot\phi)(\eta) = \phi(\gamma^{-1}\eta)$.

The Γ-space X_0 has a tautological continuous μ-harmonic function: $f(\phi)=\phi(e)$. This is, in fact, μ-harmonic since

$$\sum_{\gamma\in\Gamma} f(\gamma\cdot\phi)\mu(\gamma) = \sum_{\gamma\in\Gamma}\phi(\gamma^{-1})\mu(\gamma) = \sum_{\gamma\in\Gamma}\phi(\gamma)\hat{\mu}(\gamma) = \phi(e) = f(\phi).$$

Furthermore, since $f(\gamma\phi)=\phi(\gamma^{-1})$, f is not constant on $\Gamma\cdot\phi$ if ϕ itself is not a constant function. This simple remark suggests the following approach to finding examples of non-Liouville actions on a Γ-space S. Suppose we can somehow construct a continuous, Γ-equivariant map $\Phi: S\to X_0$. We express equivariance by $\Phi(\gamma s)=\gamma\Phi(s)$ and write $\Phi(s)=\phi_s$. If Φ does not map S entirely into the space of constant functions, then $f\circ\Phi$ defines a continuous, μ-harmonic function on S which is not Γ-invariant. Thus, what we have shown in Theorem 1.1 amounts to the following.

Proposition 6.1 *If μ is a symmetric probability measure on Γ, then every Γ-equivariant continuous $\Phi: S^1\to X_0$ maps into the constant functions.*

We now construct a continuous Γ-equivariant map $\Phi : S^2 \to X_0$ whose image is not contained in the space of constant functions. By the above remark, this yields a non-Liouville action on S^2. The example generalizes one given in [FZ2].

Let Γ and $\hat{\mu}$ be such that the Poisson boundary of $(\Gamma, \hat{\mu})$ can be identified with the circle S^1. For example, as already noted, if Γ is a uniform group of isometries of the Poincaré disc, it is possible to find a symmetric μ for which the Poisson boundary of the group coincides with the geometric boundary of the disc [Fur]. Let ν be a harmonic measure on S^1 as in the discussion immediately after Proposition 3.1. The following general fact was pointed out to us by Kaimanovich.

Lemma 6.2 *Let $\hat{\mu}$ be a non-degenerate probability measure on Γ such that a Poisson boundary (B, ν) of $(\Gamma, \hat{\mu})$ is non-trivial. Then the $\hat{\mu}$-harmonic measure ν has no atoms.*

Proof. Suppose for a contradiction that ν does have atoms, and let $b \in B$ be such that $\nu(b) \geq \nu(b')$ for all $b' \in B$. Since ν is $\hat{\mu}$-harmonic and $\hat{\mu}$ is non-degenerate, the maximum principle applied to $\gamma \mapsto (\gamma_* \nu)(b)$ implies $\nu(b) = \nu(\gamma b)$ for all $\gamma \in \Gamma$. On the other hand, if $(\gamma_1, \gamma_2, \dots)$ is a random walk on $(\Gamma, \hat{\mu})$, then $(\gamma_1 \dots \gamma_n)_* \nu$ converges to the Dirac measure δ_{g_∞}, where g_∞ is the random point on the boundary to which the random walk converges. Writing $g_n = \gamma_1 \dots \gamma_n$, then $0 < \nu(b) = (g_n)_* \nu(b) \to \delta_{g_\infty}(b)$. This is only possible if sample paths converge to b almost surely, contradicting the assumption that the Poisson boundary is non-trivial. $\square$

We now construct a non-trivial Γ-equivariant continuous map $\Phi : S^2 \to X_0$. Let $x = (z, \theta)$ in the cylinder $S^1 \times [0, 2\pi]$ represent the interval $I_x \subset S^1$ with endpoints z and $ze^{i\theta}$. Naturally, $\gamma I_x = I_{\gamma x}$. The circle boundaries $S^1 \times \{0\}$ and $S^1 \times \{2\pi\}$ are invariant under Γ and correspond to intervals of length 0 or 2π. The quotient $(S^1 \times [0, 2\pi])/\sim$ obtained by collapsing the boundary circles to points is homeomorphic to S^2 and the Γ-action on the cylinder defines an action on the quotient by homeomorphisms fixing two points, denoted $N = (S^1 \times \{2\pi\})/\sim$ and $S = (S^1 \times \{0\})/\sim$. This defines S^2 as a topological Γ-space.

Let ν be a harmonic probability measure on S^1, which is granted since S^1 is a Poisson boundary of $(\Gamma, \hat{\mu})$. Define $\Phi(x) = \phi_x$, where $\phi_x(\gamma) = \gamma_* \nu(I_x)$. Note that Φ is well-defined since $\nu(I_x)$ is 0 on all intervals of length 0, and 1 on all intervals of length 2π. A simple calculation shows that ϕ_x is $\hat{\mu}$-harmonic for each $x \in S^2$ and that Φ is Γ-equivariant. Furthermore, Φ is continuous due to the fact that ν has no atoms. Since the Poisson boundary is non-trivial, Φ does not map into the set of constant functions. This shows that (S^2, Γ, μ) is non-Liouville.

In this example, the only Γ-orbits in S^2 on which $\Phi(x)$ is a constant function are the fixed points N, S, so $\mathcal{C} = \{N, S\}$, where $\mathcal{C}$ is the set defined in Proposition 3.1. The same argument used in Lemma 5.2 shows that $g = f \circ \Phi$ can be composed with an affine transformation of $\mathbb{R}$ to insure $g(S) = 0$ and $g(N) = 1$, after which $g(x)$ is the probability that random walk on (S^2, Γ, μ) starting at $x \in S^2$ converges to N.

When (S^1, ν) coincides with the boundary of the hyperbolic disc, where Γ is a uniform lattice in $PSL(2, \mathbb{R})$, Lemma 4.2 of [FZ2] can be used to show that the just constructed action on S^2 is ergodic relative to the smooth measure class on S^2. This dynamical property of the action contrasts with the simple behavior of the random walk, which converges to N or S with probability 1.

It is interesting to regard the induced random walk on the space X_0 itself, for a general $(\Gamma, \hat{\mu})$. Let $I = [-1, 1] \subset X_0$ represent the subspace of constant functions in X_0. Then the random walk on (X_0, Γ, μ) converges towards I, i.e., the random walk on $X_0 \setminus I$ is transient. On the other hand, the Γ-action on X_0 can be dynamically very complicated. For example, it is shown in [FZ2] for subgroups of $PSL(2, \mathbb{R})$ that the $\mathbb{Z}$-action on X_0 induced by a parabolic or hyperbolic element is a chaotic dynamical system.

References

[DK] B. Deroin and V. Kleptsyn. *Random conformal dynamical systems*, GAFA, Vol. 17 (2007) 1043-1105.

[DKN] B. Deroin, V. Klepstyn, A. Navas. *Sur la dynamique unidimensionnelle en régularité intermédiaire*, preprint, 2006.

[FFP] S. Fenley, R. Feres, K. Parwani. *Harmonic functions on* $\mathbb{R}$*-covered foliations*, 2008 (to appear).

[FZ1] R. Feres and A. Zhegib. *Leafwise holomorphic functions*, Proc. Amer. Math. Soc. 131 (2003), no. 6, 1717–1725.

[FZ2] R. Feres and A. Zhegib. *Dynamics on the space of harmonic functions and the foliated Liouville problem*, Ergodic Theory Dynam. Systems 25 (2005), no. 2, 503–516.

[Fu] A. Furman. *Random Walks on Groups and Random Transformations*, Handbook of Dynamical Systems, Vol. 1A, Chapter 12. Hasselblatt and Katok, North-Holland, 2002, 931-1014.

[Fur] H. Furstenberg. *Random walks and discrete subgroups of Lie groups*, 1971 Advances in Probability and Related Topics, Vol. 1 pp. 1-63, Dekker, New York.

[Gar] L. Garnett. *Foliations, the ergodic theorem and Brownian motion*, J. Funct. Anal. 51 (1983), 285-311.

[Pa] F. Paulin. *Analyse harmonique des relations d'équivalence mesurées discrètes*, Markov Proc. Rel. Fields. 5 (1999) 163-200.

DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY, ST. LOUIS, MO 63130 USA
E-mail address: feres@math.wustl.edu; emily@math.wustl.edu