From billiards to thermodynamics

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Engraving from Charles Cotton’s 1674 book, *The Complete Gamester*
Mathematical billiards: simple models of mechanical systems that
- help to explore the foundations of statistical mechanics,
- used to develop the mathematical theory of dynamical systems
- has deep connections with many areas of mathematics.
We don’t think of it as a game of skill ... 

... but a game of observation. We set the ball in motion and try to understand what happens to it over long periods of time and how what it does is affected by the shape of the table.

Very different long term behavior of trajectories for different shapes. The example on the left shows fairly regular behavior. The one on the right is very unstable and “chaotic.” The stadium billiard is said to be ergodic.
What if there are many balls?

Single particle system in dimension 2 describes two-particle system in dimension 1. This idea applies to any number of particles in two or three dimensions.
A linear change of coordinates makes the total kinetic energy proportional to the square of the length of a vector. This makes collisions mirror-like. This idea holds in any dimension, for any number of particles.
Example: a multi-particle scattering process

Mass $m$ on the right can move freely along a rail track, while masses $m_i$ on the left can only move freely over the interval $[0, l]$. We want to understand the motion of $m$ as it approaches the wall-bound masses, collides with the $m_i$ multiple times, then eventually turns around and moves away to the right.
A billiard representation of the scattering process

The multi-particle system can be represented by a single particle billiard system in dimension $k + 1$. The vertical axis $x_0$ gives the position of $m$. ($x_i := \sqrt{m_i}|z_i|$.)
What can we say about the motion of $m$?

We add a simple, reflecting wall on the right.

The motion of $m$ is described by the projection to the axis of the rail track of $m$. This is a projection from an $(k+1)$-dimensional space to a 1-dimensional space.
Temperature comes into play

Theorem

Suppose that:

- the $m_i$ have random initial velocities with mean 0 and variance $\sigma_i^2$ such that
  \[ \beta = 1/(m_i\sigma_i^2) \]
  is the same for all $i$;
- the polygonal billiard in dimension $k + 1$ is ergodic;
- $k$ is very large;
- $m$ is small relative to the $m_i$.

Let $v_1, v_2, v_3, \ldots$ be the velocities of $m$ as it emerges from the interaction zone after each scattering event. Then the long term distribution of these velocities is approximately given by:

\[ \varrho(v) = \beta mv \exp \left( -\beta \frac{mv^2}{2} \right). \]
\( \varrho(v) \) is the *Maxwell-Boltzmann* distribution of velocities of \( m \).

Equality of \( \beta \) for \( m \) and the \( m_i \) characterizes stationarity of the iterated scattering process.

In physics textbooks

\[ \beta = \frac{1}{\kappa T} \]

where \( \kappa \) is called the Boltzmann constant and \( T \) is the absolute temperature (of the wall).

At equilibrium, the wall and \( m \) have the same temperature! (Thermal equilibrium.)

The proof of the theorem is geometric. This is a theorem about geometry in very high dimensions!
On to building a heat engine!

But first, we wish to simplify the wall heat bath/thermostat.

Deterministic version:

Probabilistic version:

Assume that prior to each scattering of $m_2$, the velocity of $m_1$ is chosen randomly, with mean 0 and variance $\sigma^2$. 
Approach to thermal equilibrium

This probabilistic model of collision can be iterated to show how thermal equilibrium is reached.
Two temperatures and heat flow
The heat flow $Q$ is the mean energy transferred from the hot to the cold wall per collision with $m_2$.

\[ Q = C(T_{\text{hot}} - T_{\text{cold}}). \]
A minimalistic heat engine

![Diagram of a minimalistic heat engine with labels for various components and angles.]
The engine’s billiard representation

The billiard representation for the heat engine. The evolution of the engine is described by an accelerating billiard particle inside this channel.

Side view of one trajectory.
Left: position of the Brownian particle (zero load), as function of time. Parameters: mass of thermostat wall $m_0 = 10$, Brownian particle $m_1 = 100$, gas molecule $m_2 = 1$. Length of circular track $l = 10^{-4}$, the vertical axis measures translation along the track. Number of events $N = 10^6$. Temperatures, from middle graph to the top: $\sigma_1^2 = 1$ and $\sigma_2^2 = 1, 2, 4, 8$. For lower graphs the two parameters are reversed.
The working engine with a force load. The graphs show the mean velocity of the Brownian particle as a function of time. Force load = 1. From top to bottom: $\sigma_1^2 = 1, 2, 4, 8$. 
The *efficiency* of the engine is the ratio

\[
\frac{\text{mean mechanical work done by the engine (rotation against load)}}{\text{heat flow from hot source}}
\]

Efficiency. (Not much to brag about... but positive!)
Nicolas Léonard Sadi Carnot (1 June 1796 - 24 August 1832). In his 1824 book *Reflections on the Motive Power of Fire*, he gave the first successful theoretical account of heat engines (Carnot cycle), laying the foundations for the second law of thermodynamics. (Wikipedia)