Math 312 - Spring 2018 - HW 3
Due February 27

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For numerically solving ODEs in this course I suggest using the package deSolve in R. (You are free to use anything also, but I'll use it in my solutions to homework exercises.) A good reference is the text *Solving Differential Equations in R*, by K. Soetaert, J. Cash, and F. Mazzia. The Olin Library has the pdf file of the book, which you may freely download.

1. Read pages 41-45 of *Solving Differential Equations in R* and reproduce the graphs on page 46 for the Lorenz model.

2. (See the numerical problem on page 156, chapter 7 of the HSD textbook.) Consider the non-autonomous (i.e., time dependent) differential equation
$$\frac{dy}{dt} = e^t \sin y.$$
   (a) Express this non-autonomous differential equation as an autonomous system of two equations in the variables x and y, where $x(t) = t$.

   (b) Draw a careful sketch of the vector field defined by the system of part (a) over the region $-2 \le x \le 2$, $-3\pi/2 \le y \le 3\pi/2$. Show on your graph the nullclines (these are the trajectories that satisfy $dy/dx = 0$.)

   (c) Draw by hand a few solutions (trajectories) of the system's phase portrait. (As you do, think about possible issues of precision that may arise in solving the differential equation numerically, using for example Euler's method, due to the sensitive dependence on initial conditions. It is not necessary to write down your thoughts.)

3. An $n \times n$ real valued matrix A is said to be *orthogonal* if $A^t A = AA^t = I$. That is to say that the transpose of the matrix is its inverse: $A^t = A^{-1}$. The orthogonal matrix is called a *rotation* matrix if its determinant equals 1. The set of all rotation matrices in dimension n constitutes a group of transformation of $\mathbb{R}^n$. This simply means that the product of two rotation matrices is a rotation matrix, the inverse of a rotation matrix is also a rotation matrix, and the identity matrix belongs to this set. The group of rotation matrices in dimension n is denoted $SO(n)$.

   (a) Let $A \in SO(3)$ be any rotation in dimension 3. Show that A has an axis of rotation (that is, a non-zero vector $u \in \mathbb{R}^3$ such that $Au = u$), and that A rotates the plane perpendicular to u by some angle θ.

   (b) Let $A(t)$ be a differentiable path in $SO(n)$ such that $A(0) = I$ (the path starts at the identity matrix). Show that $A'(0)$ is a skew-symmetric matrix. (A matrix Z is *skew-symmetric* if its transpose is the negative of itself.)

   (c) Let Z be a skew-symmetric matrix in dimension n. Show that $A(t) = e^{tZ}$ is in $SO(n)$ for all t. (In this sense, we say that skew-symmetric matrices are *infinitesimal generators* of rotations.) Also check the identities: $A'(t) = ZA(t) = A(t)Z$.

   (d) Let $u = (a, b, c)$ be a vector in $\mathbb{R}^3$ and define the linear transformation $L_u v = u \times v$ (where $\times$ is the ordinary cross-product from Calculus III.) Find the matrix $Z(u)$ that represents L_u in the standard basis of $\mathbb{R}^3$. Described geometrically the linear transformation of $\mathbb{R}^3$ represented by the matrix $A(t) = e^{\omega t Z(u)}$, where here u is a unit length vector and ω and t are real numbers.

(e) Given $Z(u)$ and $Z(v)$ as defined in the previous part, express the matrix commutator defined by

$$[Z(u), Z(v)] = Z(u)Z(v) - Z(v)Z(u)$$

in terms of $Z(u \times v)$.

(f) Let $X(t)$ be a differentiable path in $\mathbb{R}^3$ that satisfies the (non-autonomous) differential equation

$$X'(t) = Z(t)X(t).$$

Show that $X(t)$ must remain at all times on the surface of a sphere of radius $\|X(0)\|$. (This holds in any dimension, not only 3.)

4. Consider the Newtonian mechanical system shown in the figure. A rigid planar body consists of a large disc of radius R and mass M having uniform mass distribution rigidly connected to a small body of mass m (regarded as a point) by a thin rod of negligible mass and length (from the mass m to the center of the large disc) l. The whole system rotates freely, without friction, about the center of the disc.

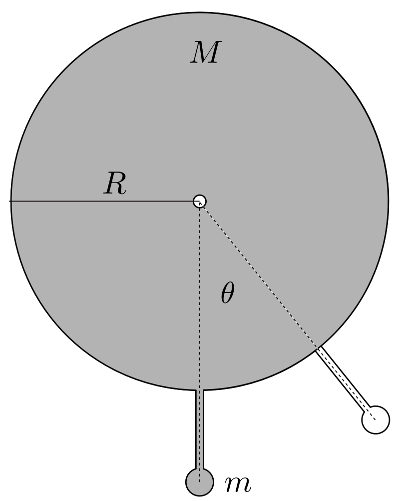

Figure 1: Variant of the simple pendulum.

(a) Write the total kinetic energy K of the system as a function of θ and $\dot{\theta}$.

(b) Write the potential energy U of the system due to gravity, as a function of θ. (Due to the circular symmetry of the large disc, only the contribution of m to the potential energy will matter.)

(c) Find the Euler-Lagrange equation. Recall that if the Lagrangian is $L(\theta, \dot{\theta}) = K(\theta, \dot{\theta}) - U(\theta)$ (the kinetic minus the potential energy) then the Euler-Lagrange equation is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$.

5. (Textbook, Exercise 1 of Chapter 7.) Write out the first few terms of the Picard iteration scheme for each of the following initial value problems. Where possible, find explicit solutions and describe the domain of this solution.

(a) $x' = x + 2;\ x(0) = 2$

(b) $x' = x^{4/3};\ x(0) = 0$

(c) $x' = x^{4/3};\ x(0) = 1$

(d) $x' = \cos x;\ x(0) = 1$

(e) $x' = 1/(2x); x(1) = 1$

6. (Textbook, Exercise 2 of Chapter 7.) Let A be an $n \times n$ matrix. Show that the Picard method for solving $X' = AX,\ X(0) = X_0$ gives the solution $\exp(tA)\,X_0$.