Math 312 - Spring 2018 - HW 4
Due April ??

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1. *More on the Picard approximation.* Consider the non-autonomous linear initial value problem

$$X' = A(t)X$$

where $A(t)$ is an $n \times n$ matrix depending continuously on t and $X(0) = X_0 \in \mathbb{R}^n$. Recall that the Picard approximation is based on the following idea. First we restate the initial value problem for the matrix-differential equation as an integral equation:

$$X(t) = X_0 + \int_0^t A(s)X(s)\,ds.$$

Then we define a sequence of functions $U_k(t)$, expected to approximate $X(t)$ as k tends to infinity, by

$$U_{k+1}(t) = X_0 + \int_0^t A(s)U_k(s)\,ds.$$

When $A(t)$ is constant (or when $A(t)$ and $A(t')$ commute for different t and t'), $U_k(t)$ is the partial sum up to order k of the Taylor series of the matrix exponential, and the solution to the initial value problem is given in terms of the exponential of A, as we have already seen. In the general case where $A(t)$ is not constant, it is still possible to express the solution in terms of something that formally resembles a matrix exponential function. To define this concept, we introduce the operation

$$\mathscr{T}[A(t_1),\dots,A(t_k)] = A(t_{i_1})\cdots A(t_{i_k})$$

where $t_{i_1} \geq t_{i_2} \geq \cdots \geq t_{i_k}$. For example, if $k = 2$,

$$\mathscr{T}[A(t_1),A(t_2)] = \begin{cases} A(t_1)A(t_2) & \text{if } t_2 \leq t_1 \\ A(t_2)A(t_1) & \text{if } t_1 \leq t_2. \end{cases}$$

Show that

$$U_k(t) = X_0 + \int_0^t A(s)X_0\,ds + \frac{1}{2}\int_0^t\int_0^t \mathscr{T}[A(s_1),A(s_2)]X_0\,ds_1\,ds_2 + \cdots + \frac{1}{k!}\int_0^t\cdots\int_0^t \mathscr{T}[A(s_1),\dots,A(s_k)]X_0\,ds_1\dots ds_k.$$

The limit of this sequence is sometimes written as $\mathscr{T}\left[e^{\int_0^t A(s)\,ds}\right]X_0$ where $\mathscr{T}\left[e^{\int_0^t A(s)\,ds}\right]$ is called the *time-ordered exponential.*

2. *Paper-rock-scissors game.* Consider the following system of chemical reactions. (This is a variant of the "Paper-

Rock-Scissors" game.) The species are $\{P, R, S\}$ and the transitions are $\{t_1, t_2, t_3\}$ where

$$t_1: P+R \rightharpoonup P+P$$
$$t_2: S+R \rightharpoonup R+R$$
$$t_3: P+S \rightharpoonup S+S.$$

(a) Draw the graph of a Petri net representing this system.

(b) Suppose that the initial state of the system is given by the triple $(1,1,1)$. This means that there are initially 1 P-token, 1 S-token and 1 R-token. What are all the possible sequences of states? (Consider all sequences of permissible transitions $t_{i_1}, t_{i_2}, \ldots$, where a transition is *permissible* if there are enough tokens of the input species to implement it.) More generally, if the initial state is (n_P, n_S, n_R), explain why there is always the possibility that the game will stop (all but one of the species go extinct) if the transitions are chosen randomly.

(c) Now assume that the numbers of tokens are very large and that it is meaningful to consider the (molar) concentrations (or fractions) of P, S, R. Using chemistry notation, I denote these concentrations by $[P], [S], [R]$, where each number is (a limit for large numbers of) the ratio of the number of tokens of each kind divided by the total number of tokens. Assume that the mass-action law holds and let k_1, k_2, k_3 be, respectively, the reaction rate constants of t_1, t_2, t_3. Write down the system of differential equations satisfied by the three concentrations. (Note, for example, that the time derivative of $[P]$ equals $k_1[P][R] - k_3[P][S]$.)

(d) Verify that the system of equations is compatible with the property that $[P]+[S]+[R]$ must be constant (equal to 1). (Show that the derivative of this sum equals zero.)

(e) We say that the system in a given state $([P], [S], [R])$ is at equilibrium if the time derivatives of the concentrations vanish at that state. Assuming that none of the concentrations is zero, show that the system has a unique equilibrium state and express it in terms of the constants k_1, k_2, k_3.

(f) Consider now the rate constants $k_1 = 2, k_2 = 1, k_3 = 3$ and initial concentrations $[P] = 1, [S] = 1, [R] = 2$. Using the computer, draw the plots of $[P], [S], [R]$ as functions of time. (Suggestion: use times from 0 to 10 in intervals of 0.01.) Also draw a 3-dimensional plot of the trajectory. You'll probably notice that the trajectory will fall on a line segment. Can you find the direction of that line segment analytically (in terms of the constants k_i)? (Not for credit, but think about it.)

3. *A model of epidemics.* The following is a simplified form of a mathematical model describing the course of an epidemic. It was proposed and analyzed by Kermack and McKendrick in the 1930's. A population that has been exposed to some contagious disease is divided into three groups: those individuals *susceptible* of contracting the disease, denoted S; *infective* individuals, I; and *removed* individuals, R. Individuals of type S may change into type I, through contagion, while those of type I may change into type R, through death, recovery, or isolation from the rest of the population. I will represent the model by the following Petri net (here the number of arrows issuing from each species represents the number of tokens used or produced by the reaction):

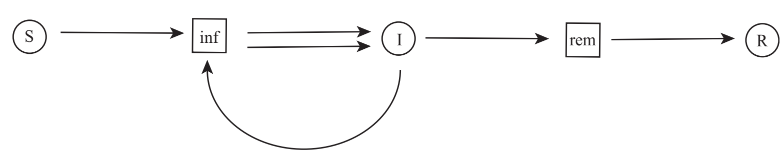

Figure 1: Petri net diagram for a model of epidemic spread.

(a) Express the model in chemistry notation by listing the set of elementary reactions.

(b) Let us assume that population numbers are large enough so that it is meaningful to regard them as continuous. Use the letters S, I, R to represent the population numbers. Suppose that the reaction inf (infection) has rate constant σ and the reaction rem (removal) has rate constant ρ and that the mass-action law holds. Write down the system of differential equations in the variables S, I, R.

(c) Making the coordinate changes

$$x = \frac{\sigma}{\rho} S,\ y = \frac{\sigma}{\rho} I,\ z = \frac{\sigma}{\rho} R,\ \tau = \rho t$$

show that the differential equations reduce to

$$\begin{aligned} x' &= -xy \\ y' &= (x-1)y \\ z' &= y. \end{aligned}$$

(d) Show that the quantities $x + y + z$ and $z + \ln x$ are conserved; that is to say, they do not change in time. Is there an equilibrium state (fixed point) for this system?

(e) Using the computer, plot the graphs of x, y, z as functions of (the rescaled) time τ assuming initial conditions $x(0) = 50, y(0) = 1, z(0) = 0$. Draw the three plots separately over time intervals that show clearly how each variable behaves. Describe in words how S, I, R evolve in time based on what your graphs show.

4. *Forced pendulum.* (Much more on this subject in "Nonlinear Oscillations and the Smale Horseshoe Map" by Philip Holmes, in Proceedings of Symposia in Applied Mathematics, American Mathematical Society, Volume 39, 1989.) Consider the simple pendulum of Figure 3. A point mass m is suspended by a rigid, massless rod of length l pivoted freely at 0 to swing in a plane. Three forces act on the bob: gravitation ($-mg$, vertically), friction or dissipation due to air resistance ($-cv$, tangentially), and the external time varying torque $\delta T(t)$ applied at the pivot. The minus signs are conventional, reflecting that the forces oppose motion as indicated, and friction is modeled by the simplest possible law: resistance is linearly proportional to speed. The state of the system is uniquely specified by the pair $\left(\theta, \frac{d\theta}{dt}\right)$, angular position and velocity. Resolving the forces in the tangential direction, and appealing to Newton's second law (force = mass × acceleration), we obtain the second order ordinary differential equation

$$\delta l T(t) - cl\frac{d\theta}{dt} - mg\sin\theta = m\frac{d}{dt}\left(l\frac{d\theta}{dt}\right). \tag{1}$$

It is possible to simplify the equation somewhat by introducing the following quantities:

$$\tau = \sqrt{\frac{g}{l}}\,t,\ \ S(\tau) = \frac{l}{mg}T(t),\ \ \gamma = \frac{c}{m}\sqrt{\frac{l}{g}}.$$

(a) Denote by $\dot\theta, \ddot\theta$ the first and second derivatives of θ with respect to the rescaled time τ. Show that the above pendulum equation (1) can be written as

$$\ddot\theta + \gamma\dot\theta + \sin\theta = \delta S(\tau).$$

(This simplification shows that the dynamics of the pendulum essentially only depends on two parameters: δ and γ, in addition to the form of the function $S(\tau)$. When δ and γ are 0 we have the classical pendulum.)

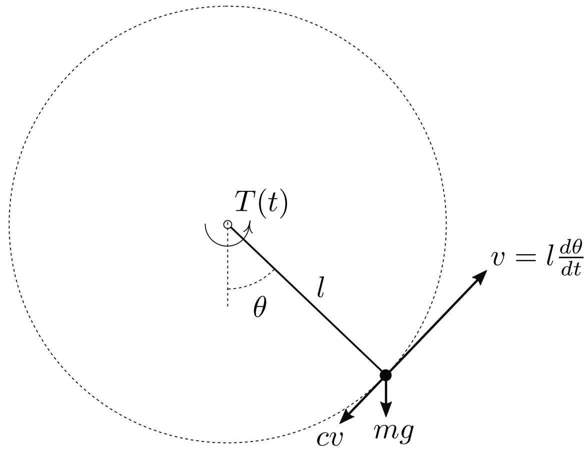

Figure 2: The simple pendulum subject to torque and friction.

(b) When $\delta = \gamma = 0$, show that the solution curves $(\theta(\tau), \dot{\theta}(\tau))$ lie on level sets of the Hamiltonian energy function (kinetic plus potential energy)

$$H(\theta, \dot{\theta}) = \frac{\dot{\theta}^2}{2} + (1 - \cos\theta).$$

(We then say that this system is *integrable*.)

(c) Introducing the new variable $v = \dot{\theta}$, and considering for concreteness the forcing term $S(\tau) = \cos(\omega\tau)$, show that the second order equation given in the above part (a) gives rise to the autonomous system of first order equations

$$\begin{aligned} \dot{\theta} &= v \\ \dot{v} &= -\sin\theta + \delta\cos(\omega\tau) - \gamma v \\ \dot{\tau} &= 1. \end{aligned}$$

(d) Later we'll study this equation analytically. For now you'll simply explore it by doing some computer experiments. Produce graphs of trajectories in the $(\theta, \dot{\theta})$ plane for the following conditions:

i. Parameters: $\delta = 1.2, \gamma = 1, \omega = 1$; initial conditions: $\theta = 0, \dot{\theta} = 0.1, t = 0$. Use time steps 0.01, starting at $t = 0$ and ending at $t = 40$.

ii. Parameters: $\delta = 1.3, \gamma = 0.03, \omega = 1$; initial condition: $\theta = 0, \dot{\theta} = -2.2, t = 0$. Use time steps 0.01, starting at $t = 0$. For these parameters, show in separate graphs orbits up to times $t = 50, 200, 500$.