Discussions of Monte Carlo Simulation in Option Pricing

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MATH 350 RESEARCH PAPER

INTRODUCTION

Having been exposed to a variety of applications of Monte Carlo methods, including entertainment, engineering and sports, we attempt to spin our interests in financial markets with a mathematical twist. In the realm of finance, the Black-Scholes model is widely applied in calculating prices of European options, which cannot be exercised before the expiration date. This model, first published in 1973 in the paper "The Pricing of Options and Corporate Liabilities", suggests a partial differential equation that governs the price of the option over time. The key idea behind the derivation was to use delta hedging the option through buying and selling the underlying asset in a logical way to mitigate risk. However, this widely acclaimed formula, which helped Scholes to win Nobel Prize in Economics, loses its magic when dealing with American options. American options differ from European style options because the right of exercise for American options is not limited to any specific point of time before maturity.

Investors are baffled by the choice of exercising time of American options because there is always uncertainty over whether the underlying assets will go higher or lower in the future. Such uncertainty creates significant complexity in evaluating the value of American options. Monte Carlo simulation is a great method to value American style options because regardless of the future price of an individual option, we should be able to derive the expected return of exercising this American option early, as long as we assume that the underlying assets' price will follow a log-normal distribution. According to our research, we find out that in 23.95% of cases, it is optimal for investors to exercise American call options early, contrary to the classical financial theory, which states that one should never exercise the call option before the maturity.

Keywords: American options, Monte Carlo simulation, options pricing, early exercise

BACKGROUND

Option pricing is essential in corporate finance decision making since many corporate liabilities can be expressed in terms of options or combinations of options and such variations help to leverage different methods. A premium is paid or received for purchasing or selling options. This price can be split into two components. *Strike price* is fixed in the contract and specifies the price at which a specific derivative contract can be exercised. The *intrinsic value* is the difference between the underlying price and the strike price, to the extent that this is in favor of the option holder. The major difference between American and European options lies in the timing of when the options can be exercised:

- American options have a more flexible exercise rule and can be exercised at any time before the expiration date; they usually trade on standardized exchanges.
- European options can be exercised only at the end of their lives, which is at its maturity or the expiration date of the option, i.e. at a single pre-defined point in time; it normally trades over the counter.

For both, the payoff—when it occurs—is via:

Call option: Max [$(S - K)$, 0]

Put option: Max [$(K - S), 0$], where K is the strike price and S is the spot price of the underlying asset.

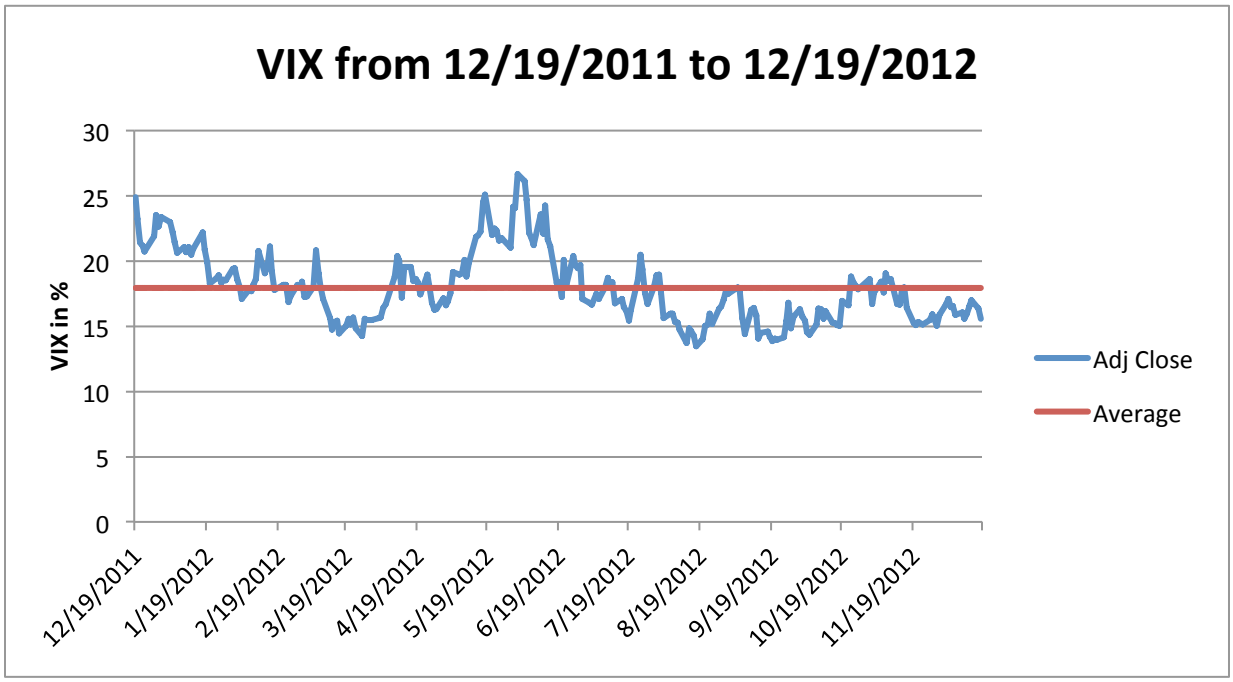
A large number of exchange-traded stock options, such as those issued for companies like Apple and Wal-Mart, are American-style options. However, financial index options can be issued as either American- or European-style options. For example, according to data from investopedia.com, S&P 100 Index options are traded as American-style options, whereas Nasdaq 100 Index options are traded as European-style options.

LITERATURE REVIEW

Upon reading a number of research papers, the Monte Carlo approach has proved to be a valuable and flexible computational tool in security pricing. Moreover, due to various reduction methods and models, there are also different ways to prove the efficiency of the Monte Carlo model. Previous literature explores a number of models of option pricing under conditions of general equilibrium as well as exploring the hedging position of logarithmic distributions of different options. Fu (2011) explains different ways to value a put and call option: for a *call option*, the option is in-the-money if the underlying price is higher than the strike price; then the intrinsic value is the underlying price minus the strike price. For a *put optio*n, the option is in-the-money if the *strike* price is higher than the *underlying* price; then the intrinsic value is the strike price minus the underlying price. Otherwise the value of the option is zero. Boyle (1997) suggests that the Monte Carlo method simulates the process of generating the returns on the underlying asset and invokes the risk neutrality assumption to derive the value of the option. Fu (2011) also explains several primary methods for pricing American- style options, such as binomial trees and other lattice methods (i.e. trinomial trees), and finite difference methods to solve the associated boundary value partial differential equations (PDEs). However, some of these computational methods can only handle one or two sources of uncertainty, which creates some limitation when it comes to evaluating multiple sources of uncertainty.

DATA COLLECTION

Different securities usually do not share the same characteristics, which further complicates the valuation method. To make the scope of the research feasible within the time



frame that we have in class, we will have to narrow our analysis down to a specific type of underlying assets. After extensive research and comparisons, we chose the Standard & Poor 500 index (Ticker symbol: ^GSPC)[1] as our sample to approximate the equity price. S&P 500 is an index of 500 stocks chosen for market size, liquidity and industry sector. It is an appropriate sample for our study not only because S&P 500 is a leading indicator of U.S. equities, but also because it reflects the risk/return characteristics of the large cap universe (equities that are valued at more than $10 billion). In order to measure the standard deviation of S&P 500, we used VIX, the Chicago Board Options Exchange Market Volatility Index, which is designed by the Chicago Board Option Exchange to gauge the implied volatility of S&P 500 each day. By checking

[1] http://www.marketwatch.com/investing/index/spx/options (Data assessed on Dec 19th 2012)

Wolfram Alpha, we also know that the average annual return for the S&P 500 is 9.07% in the last five years.

RESEARCH METHODOLOGY

Though most path-dependent featured options are easily priced through the simulation of sample paths, pricing American style options generally requires a backward algorithm. Through estimating backwards from the exercise date of the option via dynamic programming, the optimal strategy and option price can be estimated. We use American options expiring one year from today (Dec 19th 2012), within which there are 252 trading days. This means that one simulation consists of 252 steps. Within each step there are two potential movements: up by $e^{\left(r-\frac{\sigma^2}{2}\right)*h+\left((\sigma*\sqrt{h}*\epsilon)\right)}$ or move down by $\frac{1}{e^{\left(r-\frac{\sigma^2}{2}\right)*h+\left((\sigma*\sqrt{h}*\epsilon)\right)}}$.

This function is derived from the property of lognormal distribution which implies that

$$E\left(\frac{S_{t+h}}{S_t}\right) = e^{\left(r-\frac{\sigma^2}{2}\right)*h+\left(E\left(\sigma*\sqrt{h}*\epsilon_t\right)+\frac{1}{2}*Var\left(\sigma*\sqrt{h}*\epsilon_t\right)\right)} = e^{rh}$$

After one simulation is finished, we will determine the number of times it is optimal for investors to exercise the option early. If N simulations are finished, there should be in total 251*N times in which investors can exercise the option early. Using it to divide the actual number of times an investor may optimally engage in early exercise, we can get the percentage of time that an investor should exercise an American option early.

SIMULATION PROCESS

1. Create all necessary variables, including total number of simulations *N*, number of trading days each year *n*, expected return for S&P 500 using average of the five year realized annual return *r*, volatility *sigma*, index value at day 0 (today) *iniIndex*, strike price *Strike*, risk free rate *rf* and all matrices we will need to store values.
2. Within each of the 252 steps for one simulation, we will generate a random number with normal distribution first. The variable "change" is calculated using the aforementioned equation.
3. In order for investors to be indifferent between holding the S&P 500 and another asset generating 9.7% return as well, the probability for S&P 500 to go up is calculated as $\frac{9.07\%-down}{up-down}$ in which up is defined as "change" when "change" is greater than 1 and down as "1/change", vice versa.
4. A random number is generated to determine if the index value will go up or down.
5. Repeat step 2-4 for the rest of the 251 steps.
6. Repeat step 2-5 for the rest of the N-1 simulations.
7. After finishing the simulation, we essentially have the performance of the S&P 500 in 10000 different worlds. To determine when to exercise early, we will look at the index number at each step and calculate its future value. If the result is higher than the ending index number of that specific world, we call that step as an optimal early exercise time.
8. Divide the total number of optimal exercise opportunities by (n-1)*N to get the percentage amount of time investors will be better off by exercising their American call option early.

CONCLUSION

American call option with financial theory

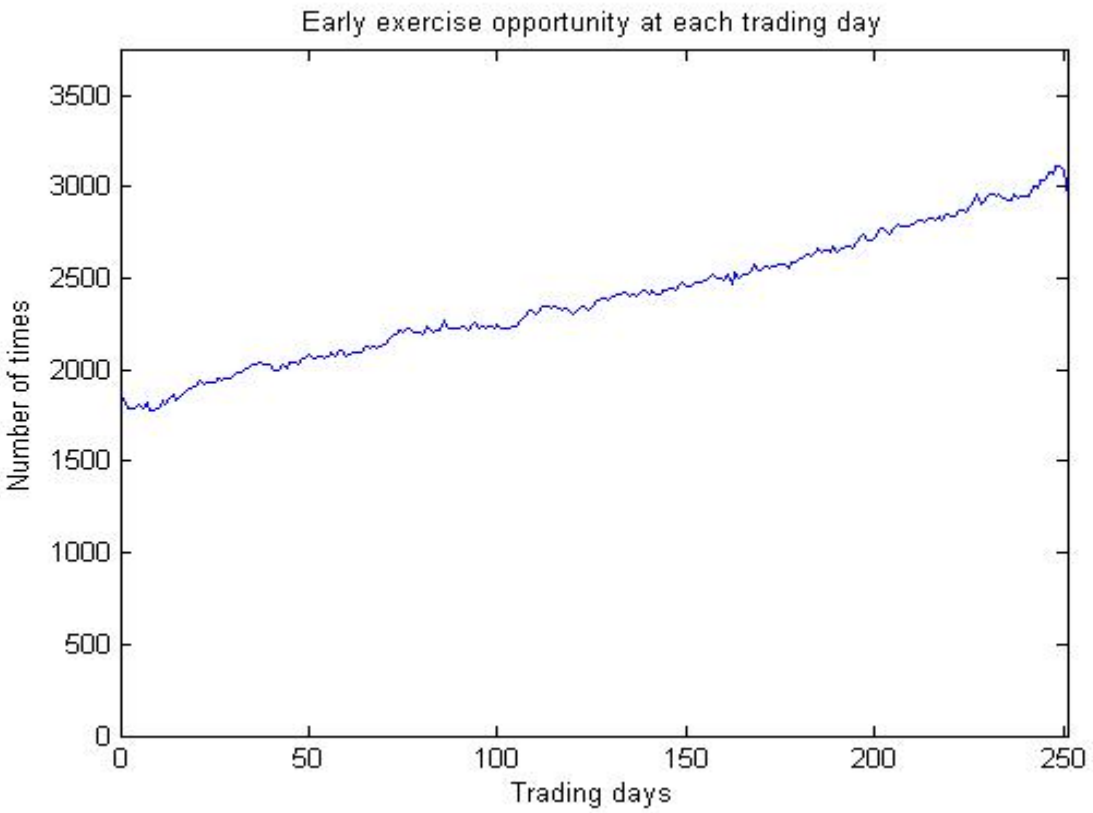
According to classical financial theory, for American-style call options without cash dividends, the option should never be exercised before the maturity date. There are a number of reasons to explain this theory. First, for a given movement, i.e. upward or downward, in the price of an underlying asset, the profit from holding an in-the-money call is equivalent to the profit from holding the underlying asset. The call option, nevertheless, has the added benefit of protecting against the risk of a downward price movement below the strike price. Moreover, because of the discount rate, it costs more to exercise the option today at a fixed strike price P than to exercise it in the future at P. Last but not least, there is an intrinsic time value of the option that would be lost by exercising the option prior to the expiration date.

Our findings on early exercise of American call option

As our program shows, in 23.95% of the time, it is optimal for investors to exercise their American call option early, even after taking into account the time value of money. This surprisingly large value suggests that in the market the right to exercise early does add value to American call options, even when there is no dividend. It explains the discrepancies between prices for American call options and European call options on high-tech companies that rarely pay dividends.

The plot below shows the number of optimal early exercises for each of the 252 trading days for 10000 simulations. Its shape aligns with our prediction that early exercise gets more

appealing to investors when an American call option approaches its expiration date and uncertainty about the future index values decreases.



Finding on exercise of American Put option

We applied our algorithm on American put options by just reversing the sign of part of the program. The result suggests that investors should exercise their put option early 80% of the time. However, this is not representative of the true overall picture for two reasons. First, put options are widely used as a hedging instrument. Other than the monetary gain investors may generate by exercising the option early, there are real option values by holding onto it, such as reduced cost of financial distress. Moreover, put options, comparing to calls, are subject to greater changes when volatility increases. The time window will rapidly vanish as time elapses.

FURTHER DISCUSSION

We have to point out that despite knowing the percentage amount of time at which early exercise is optimal, the exact date still remains elusive. Without knowledge of higher-level mathematics, such as dynamic simulation, there is no easier way to predict a precise date to exercise early. Should there be more time, we would like to assign a 23.95% chance of early exercise to each trading day regardless of the index value at that day, and calculate the expected payoff for doing so compared to the payoff of exercising at the ending date. Comparison of these two values should give us a clearer picture of the added value provided to American call options by the right of early exercise.

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APPENDIX: MATLAB PROGRAM

```
N=10000;
%Total number of simulations
n=252;
%252 trading days for one year
r=0.0907;
%Realized annual for the last 5 years
h=1/n;
%Each trading day therefore is 1/252 year;
sigma=0.17937;
%Average VIX for the last one year
iniIndex=1439.66;
%S&P 500 Index as of 12/19/2012
Strike=1439.66;
%Set strike price to a certain value;
rf=0.0016;
%One year treasury rate
indexValue=zeros(N,n);
ee=0;
%Set the early exercise count to 0;
eeDate=zeros(1,n-1);
%Set the early exercise date to 0;

for o=1:N
    for i=1:n
        epsilon=randn();
        %Random number with normal distribution
        change=exp((r-(sigma^2)/2)*h+(sigma*sqrt(h)*epsilon));
        %Expected index value change
        if change>1;
            up=change;
            down=1/change;
        else
            down=change;
            up=1/change;
        end;

        Prob=(exp(r*h)-down)/(up-down);
        %Calculate prob with r, h, up and down
        x=rand;

        if i==1;
            if x<Prob
                indexValue(o,i)=iniIndex*up;
            else
                indexValue(o,i)=iniIndex*down;
            end;
        else
            if x<Prob
                indexValue(o,i)=indexValue(o,i-1)*up;
            else
                indexValue(o,i)=indexValue(o,i-1)*down;
            end;
        end
```

```
    end;
    for j=1:n-1
        if indexValue(o,j)*exp(rf*h*(n-j))>indexValue(o,n)
            %Check if index value at one specific point is higher than the
            %ending value;
            if indexValue(o,j)>Strike;
                %Check if index value at one specific point is higher than
                %the strike price;
                ee=ee+1;
                %Early exercise count +1
                eeDate(1,j)=eeDate(1,j)+1;
                %Early exercise date +1
            end
        end
    end;
end;

eeRatio=ee/((n-1)*N)
%Early exercise ratio;
plot(1:n-1,eeDate);
xlabel('Trading days');
ylabel('Number of times');
title('Early exercise opportunity at each trading day');
%Plot the optimal days within a year for early exercise
axis([0 n-1 0 1.2*max(eeDate)]);
```

PROGRAM OUTPUT

```
eeRatio =

    0.2395
```

PLOTS

Random selections of S&P 500 performance from the 10000 simulations: