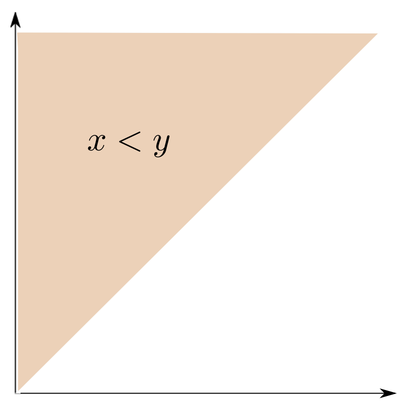
Math 350 - Homework 2 - Solutions

1. If X and Y have a joint probability density function given by

$$f(x,y) = 2e^{-(x+2y)}$$

for x and y in $(0,\infty)$, find the probability $P(X < Y)$.

The probability density function $f(x,y)$ is defined on the first (positive) quadrant of $\mathbb{R}^2$. The Event $\{X < Y\}$ corresponds to the subset of the plane described by the edge-shaped region shown in the figure.



The probability of this event is then given by the integral of $f(x,y)$ over that region. I.e.,

$$P(X < Y) = \int_0^\infty \int_0^y 2e^{-(x+2y)}\,dx\,dy = 2\int_0^\infty e^{-2y}\left(\int_0^y e^{-x}\,dx\right)dy = \frac{1}{3}.$$

2. The continuous random variable X has a probability density function given by

$$f(x) = cx$$

for $0 < x < 1$. Find the expected value $E[X]$. (You need to determine the value of c.)

The value of c comes from the condition $\int_0^1 f(x)\,dx = 1$. This integral is $c/2$, so $c = 2$. The expected value is then

$$E[X] = \int_0^1 2x^2\,dx = \left[\frac{2x^3}{3}\right]_0^1 = \frac{2}{3}.$$

3. An airplane needs at least half of its engines to safely complete its mission. If each engine independently functions with probability p, for what values of p is a three-engine plane safer than a five-engine plane?

Under the assumptions of the exercise, the probability that j engines amongst n will work fine through the duration of the mission is given by the binomial probability

$$p_{n,j} := \binom{n}{j} p^j (1-p)^{n-j}.$$

Thus we have that the probability that a 3 engine airplane will run its mission safely is

$$p_{3,3} + p_{3,2} = \binom{3}{3} p^3 (1-p)^0 + \binom{3}{2} p^2 (1-p)^1 = p^3 + 3p^2(1-p).$$

The corresponding probability for a 5 engine plane is

$$p_{5,5} + p_{5,4} + p_{5,3} = \binom{5}{5} p^5 (1-p)^0 + \binom{5}{4} p^4 (1-p)^1 + \binom{5}{3} p^3 (1-p)^2 = p^5 + 5p^4(1-p) + 10p^3(1-p)^2.$$

The values of p for which a 3 engine plane is safer than a 5 engine plane are those for which the first probability is greater than the second. I.e., we need to solve the following inequality for p:

$$p^3 + 3p^2(1-p) > p^5 + 5p^4(1-p) + 10p^3(1-p)^2.$$

This can be simplified to

$$2p^3 - 5p^2 + 4p - 1 < 0.$$

Some guesswork (informed by plotting the graph of the cubic function) gives that 1 and 1/2 are roots of the polynomial. Using the full factorization of the polynomial, we can write the inequality as follows:

$$(p-1)^2(2p-1) < 0.$$

But this quantity is negative exactly when the term $2p-1$ is negative. Therefore, the 3 engine plane is safer exactly when

$$p < \frac{1}{2}.$$

4. If X is a Poisson random variable with parameter λ, show that

   (a) $E[X] = \lambda$.

   (b) $\mathrm{Var}(X) = \lambda$.

Before solving the problem, note the following infinite series values:

$$\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = e^\lambda$$

$$\sum_{j=0}^{\infty} \frac{j\lambda^j}{j!} = \lambda \sum_{j=1}^{\infty} \frac{\lambda^{j-1}}{(j-1)!} = \lambda e^\lambda$$

$$\sum_{j=0}^{\infty} \frac{j^2\lambda^j}{j!} = \lambda \sum_{j=1}^{\infty} \frac{j\lambda^{j-1}}{(j-1)!} = \lambda \sum_{j=0}^{\infty} \frac{(j+1)\lambda^j}{j!} = \lambda^2 e^\lambda + \lambda e^\lambda.$$

From this we immediately get

$$E[X] = e^{-\lambda} \sum_{j=0}^{\infty} \frac{j\lambda^j}{j!} = \lambda$$

and

$$\mathrm{Var}(X) = E[X^2] - E[X]^2 = e^{-\lambda}\sum_{j=0}^{\infty}\frac{j^2\lambda^j}{j!} - \left(e^{-\lambda}\sum_{j=0}^{\infty}\frac{j\lambda^j}{j!}\right)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

5. Two players play a certain game until one has won a total of five games. If player A wins each individual game with probability 0.6, determine:

   (a) what is the probability she will win the match?

   It will be helpful to introduce some notation. Let W_1 indicate the event that player 1 wins the match, and W_2 the event that player 2 wins the match. Also let N_j, for $j = 1, 2, \dots,$ denote the event that a match will be exactly j games long. Note that a match cannot be decided with fewer than 5 games, and it involves no more than 9 games. Let p denote the probability that player 1 wins a single game, and $q = 1 - p$. By assumption $p = 0.6$. We wish to find the probability $P(W_1)$.

   First observe that $W1 \cap N_j$ is the event that player 1 wins and the match lasts exactly j games. Since $P(N_j) = 0$ if $j < 5$ or $j > 9$, we have

$$P(W_1 \cap N_j) = 0 \text{ if } j < 5 \text{ or } j > 9.$$

   If $5 \leq j \leq 9$, I claim that

$$P(W_1 \cap N_j) = \binom{j-1}{4}p^5q^{j-5}.$$

   This can be explained as follows. Each elementary outcome of the event $W_1 \cap N_j$ can be represented by a vector of the form $(a_1, \dots, a_j)$, where an entry a_s is either 0 (player 1 loses) or 1 (player 1 wins). For example, $(0, 1, 0, 1, 1, 0, 0, 1, 1)$ is an element of $W_1 \cap N_9$. Notice that a vector of this kind represents an outcome in $W_1 \cap N_j$, for $5 \leq j \leq 9$, if and only if: (i) it has length j, (ii) it ends in 1, and (iii) exactly 4 of the first $j - 1$ entries are equal to 1. Thus each elementary outcome of $W_1 \cap N_j$ has probability p^5q^{j-5} and there are exactly $\binom{j-1}{4}$ of them (since the last entry must be 1). Therefore,

$$P(W_1 \cap N_j) = \begin{cases} \binom{j-1}{4}p^5q^{j-5} & \text{if } 5 \leq j \leq 9 \\ 0 & \text{if } j < 5 \text{ or } j > 9. \end{cases}$$

   Noting that $W_1 = (W_1 \cap N_5) \cup \cdots \cup (W_1 \cap N_9)$, a union of mutually exclusive events, then

$$\begin{aligned} P(W_1) &= P(W_1 \cap N_5) + \cdots + P(W_1 \cap N_9) \\ &= \sum_{j=5}^{9}\binom{j-1}{4}p^5q^{j-5} \\ &= p^5(1 + 5q + 15q^2 + 35q^3 + 70q^4) \\ &= 0.7334 \end{aligned}$$

   (b) what is the expected number of games in a match?

   The expected number of matches is

$$\sum_{j=5}^{9} jP(N_j) = \sum_{j=5}^{9} j\left(P(N_j \cap W_1) + P(N_j \cap W_2)\right) = \sum_{j=5}^{9} j\left(\binom{j-1}{4}p^5q^{j-5} + \binom{j-1}{4}q^5p^{j-5}\right).$$

Therefore,

$$\begin{aligned} E[\# \text{ of Games}] &= 5(p^5+q^5) + 30(p^5q+q^5p) + 105(p^5q^2+q^5p^2) + \\ &\quad 280(p^5q^3+q^5p^3) + 630(p^5q^4+q^5p^4) \\ &= 7.3538. \end{aligned}$$

(c) Confirm your result by doing a computer simulation of the situation. For example, you can simulate the outcome of a single game by flipping a biased coin (as in homework 1) with probability of heads equal to 0.6. Play the coin game a number of times, keeping a record of the accumulated number of heads and tails. The process stops the first time that the count of heads or tails reaches 5. (This cannot take more than 9 tosses.) Now determine which one (heads or tail) reached 5 first, and how many steps it took for that to happen. By repeating the process a large number of times (say, 1000) count the frequency of the times when heads (the first player) wins. Similarly, obtain the average number of games (coin tosses) in a match.

Here is one possible way of doing this in Matlab (recall that anything in a line following % is simply a comment):

```
%The main parameters are: p (probability that player 1 wins a game)
%and m (number of sample matches)

p=0.6;
m=100000;

%The outcome of each game is decided by flipping a biased coin
%with probability of heads equal to p. Thus we call for
%m strings of random numbers (0 or 1), with P(1)=p, each string
%of length 9. If the match is decied before step 9 we just ignore the
%remaining game outcomes.

G=(rand(m,9)<p);

%The cumulative sum of each row of G gives the number of wins of
%player 1 at each step of the match.

CS1=cumsum(G,2);

%Now consider the following quantity:

T1=sum((CS1<5),2)+1;

%For each sample match, T1 is a number such that 5<=T<=10.
%T1-1 is the number of games in a match before player 1 won a
%total of 5 games. If player 1 never made 5 wins, then T1=10.
%If T1<=9, then it must have won the match. Thus we the condition
%T1<=9 is equivalent to player 1 winning the match. The frequency of
%wins of player 1 is then
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f=sum(T1<=9)/m

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%To obtain the mean number of games in a match, we observe
%that, if player 1 wins, then T1 is the time when that happened.
%Therefore, if we define the corresponding quantity, T2, for player 2,
%the match ends at the minimum of T1 and T2.

CS2=cumsum(~G,2);
T2=sum((CS2<5),2)+1;
E=min(T1,T2);

%The average length of a match is now

l=sum(E)/m
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Here are a typical sample values of f (frequency of wins of player 1) and l (average length of a match) taking the number of trials of the simulated match to be $m = 100000$:

```
f =

    0.7322


l =

    7.3523
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