Math 350 - Homework 5

Due 2/26/2010

1. (Text, problem 4, page 63.) A deck of 100 cards—numbered $1, 2, \dots, 100$—is shuffled (i.e., a random permutation is applied to the cards in the deck) and then turned over one card at a time. Say that a "hit" occurs whenever card i is the ith card to be turned over, $i = 1, \dots, 100$. Write a simulation program to estimate the expectation and variance of the total number of hits. Run the program. Find the exact answers and compare them with your estimates.

2. This problem is about the acceptance-rejection method.

   (a) Write a program that implements the acceptance-rejection method to obtain a random variable X taking values in $\{1, 2, \dots, n\}$ with probabilities $P\{X = j\} = p_j$. Assume that the random variable Y (which is accepted or rejected to obtain X) is uniform with values in $\{1, 2, \dots, n\}$. (As input variables, take the vector of probabilities $p = (p_1, \dots, p_n)$, where n is arbitrary, and as output variable the sample value of X.)

   (b) Suppose now that $n = 4$ and $p_1 = 0.2$, $p_2 = 0.3$, $p_3 = 0.4$, $p_4 = 0.1$. Test that your program is sound by generating a sequence $X_1, X_2, \dots, X_k$, for some large k, and check that the frequency of occurrence of $j = 2$ is approximately 0.3.

3. Suppose that $p = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})$ and $q = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ are two probability vectors and $\alpha = 1/3$. Write a program based on the composition method of section 4.5 that generates a random variable X with probabilities

$$P(X = j) = \alpha p_j + (1 - \alpha) q_j.$$

   Do a similar test as in the previous problem to check that your program does what is expected. (The composition method amounts to the following: Let Y be a random variable with probability vector p and Z a random variable with probability vector q, both taking values in $\{1, 2, 3, 4\}$ in the present case. Then simulate a Bernoulli random variable B with parameter α. If $B = 1$ generate a sample value of Y and set $X = Y$; if $B = 0$, generate a sample value of Z and set $X = Z$.)

4. A (discrete time) Markov chain consists of a sequence of random variables $X_0, X_1, X_2, \dots$, not necessarily independent or equally distributed, characterized by the following properties:

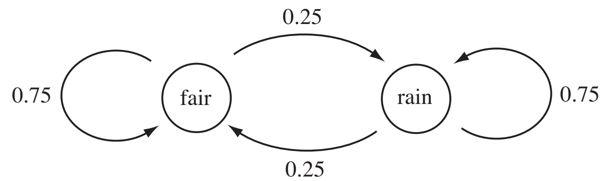
   (a) Each X_j takes values in a set $S = \{s_1, s_2, \dots\}$ (finite or infinite), which we call the *set of states* (of a system whose time evolution is being modeled by the chain);

   (b) The initial state X_0 has a probability distribution $P(X_0 = j) = \pi_j$. We call (π_j) the *initial distribution* of the chain.

   (c) For each $n = 1, 2, \dots$, the conditional probability $P(X_n = j | X_{n-1} = i) = p_{ij}$ is given. These are called the *transition probabilities* of the chain.

   As a simple example, consider the following crude model of weather forecasting. The town of Markoville has only two possible weather conditions: $s_1 =$ "fair" and $s_2 =$ "rainy". Empirical observation has shown that the best predictor of Markoville's weather tomorrow is today's weather, with the following day-to-day transition probabilities:

today \ tomorrow	rain	fair
rain	0.75	0.25
fair	0.25	0.75



For example, if today's weather is rainy, the probability that tomorrow's is fair is $p_{21} = 0.25$, and that tomorrow's is also rainy is $p_{22} = 0.75$.

Write a program that simulates Markoville's weather for the next 1000, assuming that the weather today is "fair."

5. A more general Markov chain program.

   (a) Write a general program that simulates a Markov chain $X_1, \ldots, X_n$ with the data: $\pi = (\pi_1, \ldots, \pi_k)$ the initial distributions; $P = (p_{ij})$ the transition probabilities matrix; and n, the number of random variables in the chain.

   (b) Suppose that $\pi = (0.2\ 0.5\ 0.3)$ and

$$P = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.4 & 0.1 & 0.5 \end{pmatrix}.$$

   Use your program (for a large enough n) to find the frequency of occurrence of each of the three states in the chain.

As always, I plan to discuss some of these problems in class during the week. It will help a lot if you think about them beforehand.