Math 449 Fall 2006 - Final Exam

Due 12/13/06

This exam should be done individually, without collaboration. You are not allowed to consult anyone other than your instructor. You can use your lecture notes and textbook. There is no time limit. Please turn in your solutions by December 13 the latest.

1. The differential equation for the motion of a forced pendulum is

$$ml\theta'' = -mg\sin(\theta) - cl\theta' + \delta lT(t)$$

   where m is the length of the swinging arm (whose mass we neglect), m is the mass at the end of the arm, g is the acceleration of gravity, c is a constant that accounts for friction, δ is a dimensionless parameter and $T(t)$ is a torque at the pivot, to be specified. Set $\gamma = (c/m)\sqrt{g/l}$ and $S(t) = lT(t)/(gm) = \cos(t)$. The equation then simplifies to

$$\theta'' + \sin(\theta) = \delta S(t) - \gamma\theta'.$$

   (a) We first assume that $\delta = \gamma = 0$ (no friction and no forcing term.) Using the Runge-Kutta method of order 4, solve this differential equation over the time interval $[0, 10]$ and the following initial conditions (the pairs of points are (θ, θ') at $t = 0$; there are 8 pairs):

$$(\pi/4, 0), (\pi/2, 0), (9\pi/4, 0), (5\pi/2, 0), (0, \pm 2.1), (\pm(\pi - 0.05), 0).$$

   Display the solutions for all the initial conditions on the same coordinate system on the plane (θ, θ'). (Suggested step size: $h = 0.1$.)

   (b) Describe in words what each curve corresponds to, physically.

   (c) Now, we fix the initial condition to be $(0, 1)$ and vary the parameters δ and γ. Draw separate graphs for the angular velocity θ' as function of time over the time interval $[0, 100]$ for the following parameter values: $(\delta, \gamma) = (0.1, 0.1), (0.8, 0.1), (1.5, 0.1)$. (Suggested step size: $h = 0.1$.) Is there anything notable, qualitatively, that distinguishes these graphs?

   You can create a little movie of the swinging pendulum by plotting $X = \sin(a)$, $Y = -\cos(a)$, where a is the angle, using `comet(X,Y)`. Of course, you do not need to turn this in.

2. Consider the system of differential equations:

$$\begin{aligned} x' &= \sigma(y - x) \\ y' &= rx - y - xz \\ z' &= xy - bz \end{aligned}$$

where σ, r, b are positive parameters. This system was introduced by Ed Lorentz (1963) as a drastically simplified model of convection rolls in the atmosphere. It is a famous example illustrating "chaotic motion" and some of the lessons learned from it have made into the pop culture, like the "butterfly effect," regarding sensitivity of solutions to initial conditions.

Use Runge-Kutta of order 4 to find the solution with initial condition $(1, 0, 0)$ over the time interval $[0, 50]$. Use the following parameter values: $\sigma = 10, b = 8/3, r = 28$. (Suggested step size: $h = 0.005$.) Display two graphs: one for the parametric equation of the solution $(x(t), y(t), z(t))$ (use the `plot3` command) and another for y as a function of t.

You may find it interesting to watch the evolution of the trajectory using the comet plot `comet3`. It will show clearly how the trajectory circles around two poles, every now and then moving from one to the other in a seemingly random fashion.

3. The linear wave equation (Algorithms and Problems 8, 10.1, page 556.) Solve numerically the wave equation $u_{tt}(x, t) = 4u_{xx}(x, t)$ for $0 \leq x \leq 1$ and $0 \leq t \leq 1$, with the following boundary conditions:

$$\begin{aligned} u(0, t) &= 0 && \text{for } 0 \leq t \leq 1 \\ u(a, t) &= 0 && \text{for } 0 \leq t \leq 1 \\ u(x, 0) &= \sin(2\pi x) + \sin(4\pi x) && \text{for } 0 \leq x \leq 1 \\ u_t(x, 0) &= 0 && \text{for } 0 \leq x \leq 1 \end{aligned}$$

Use the values $h = 0.05$ and $k = 0.01$ in Program 10.1, page 554. Use the `surf` and `contour` commands to plot the approximate solution.

4. The linear diffusion equation. (Algorithms and Programs 3, page 567.) Consider the equation $u_t(x, t) = c^2 u_{xx}(x, t)$, for $0 < x < 2$ and $0 < t < 0.5$, with the initial condition $u(x, 0) = \sin(\pi x) + \sin(2\pi x)$ for $0 \leq x \leq 2$, and the boundary conditions

$$\begin{aligned} u(0, t) &= t^2 && \text{for } x = 0 \text{ and } 0 \leq t \leq 0.5 \\ u(2, t) &= e^t && \text{for } x = 2 \text{ and } 0 \leq t \leq 0.5. \end{aligned}$$

   (a) Modify program 10.3, page 565, to accept the boundary conditions $u(0, t) = g_1(t) \neq 0$ and $u(a, t) = g_2(t) \neq 0$.

   (b) Solve the diffusion equation using your modified program. (Suggested step size: $h = 0.04, k = 0.01, r = 1$.) Use the `surf` command to plot the approximate solution.