Math 449 Fall 2006
Solutions to Midterm Exam

10/18/06

1. Let $f(x) = x^2 - x - 3$.

   (a) Find the Newton-Raphson formula $x_k = g(x_{k-1})$.

   (b) Start with $x_0 = 0$ and find x_1 and x_2.

   (c) Suppose that the bisection method is used to approximate the zero of $f(x)$ in the interval $[2, 3]$. How many times must this interval be bisected to guarantee that the approximation c_n has accuracy of 10^{-10}?

   Answer:

   (a) $x - f(x)/f'(x) = x - (x^2 - x - 3)/(2x - 1) = (x^2 + 3)/(2x - 1)$. Therefore,

   $$x_k = (x_{k-1}^2 + 3)/(2x_{k-1} - 1).$$

   (b) $x_0 = 0$, $x_1 = (0 + 3)/(0 - 1) = -3$, $x_2 = (9 + 3)/(-6 - 1) = -12/7$.

   (c) If x represents the zero we wish to approximate, then

   $$|x - c_n| \leq (3 - 2)/2^{n+1}.$$

   To insure accuracy of 10^{-10} we impose the condition

   $$1/2^{n+1} < 10^{-10}.$$

   This implies $n > 10/\log 2 - 1 = 32.22....$ Therefore, 33 bisections is enough.

2. Find the triangular factorization $A = LU$ for

$$A = \begin{pmatrix} 1 & -2 \\ 4 & 2 \end{pmatrix}$$

Answer: Write $A = L(L^{-1}A)$, where L is

$$L = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$

Define $U = L^{-1}A$. Notice that

$$U = \begin{pmatrix} 1 & -2 \\ 0 & 10 \end{pmatrix}.$$

Then $A = LU$.

3. Consider the following linear system of equations

$$\begin{aligned} 4x - y &= 15 \\ x + 5y &= 9. \end{aligned}$$

   (a) Write down the Gauss-Seidel iteration formula for this system.
   (b) Starting with $\mathbf{P}_0 = \mathbf{0}$, use this iteration to find $\mathbf{P}_1$ and $\mathbf{P}_2$.
   (c) Does the iteration converge? (Why?)

Answer:

   (a) The Gauss-Seidel iteration for this system takes the form:

$$\begin{aligned} x_{k+1} &= (15 + y_k)/4 \\ y_{k+1} &= (9 - x_{k+1})/5. \end{aligned}$$

   (b) If $\mathbf{P}_0 = (0,0)$, then $\mathbf{P}_1 = (15/4, (9 - 15/4)/5) = (15/4, 21/20)$, and $\mathbf{P}_2 = ((15+21/20)/4, (9-321/80)/5) = (321/80, 399/400)$. Therefore,

$$\mathbf{P}_1 = (3.75, 1.05)\ \mathbf{P}_2 = (4.0125, 0.9975).$$

   (c) Yes, the iteration converges. This is granted by the fact that the coefficients matrix is strictly diagonally dominant.

4. Compute the divided-difference table for the tabulated function and write down the Newton polynomial $P_2(x)$.

k	x_k	$f(x_k)$
0	4.0	2.00
1	5.0	2.24
2	6.0	2.45

Answer:

(a) The divided-difference table is as follows:

k	x_k	$f[\cdot]$	$f[\cdot,\cdot]$	$f[\cdot,\cdot,\cdot]$
0	4.0	2.00	–	–
1	5.0	2.24	0.24	–
2	6.0	2.45	0.21	-0.015

(b) The Newton polynomial is

$$P_2(x) = 2 + 0.24(x-4) - 0.015(x-4)(x-5).$$

5. Let $f(x) = x + \frac{2}{x}$. Use quadratic Lagrange interpolation based on nodes $x_0 = 1, x_1 = 2$, and $x_2 = 2.5$ to approximate $f(1.5)$.

Answer: The values of the function at the given x_k are

$$y_0 = 3, y_1 = 3, y_2 = 3.3.$$

The Lagrange interpolation polynomial takes the form:

$$\begin{aligned} P_2(x) &= 3\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + 3\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + 3.3\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\ &= 2(x-2)(x-2.5) - 6(x-1)(x-2.5) + 4.4(x-1)(x-2). \end{aligned}$$

From this we obtain $P_2(1.5) = 2.9$.

6. Let $g(x) = 1 + \frac{1}{2x}$.

(a) Start with the initial point $x_0 = 1$. Compute the points x_1 and x_2 in the fixed point iteration of $g(x)$.

(b) Explain graphically how the iteration proceeds. (Draw your graphs carefully enough to be able to decide visually whether the iteration converges.)

(c) How can we determine analytically if the iteration converges? (Hint: consider the derivative of $g(x)$ for $x \geq 1$.)

Answer:

(a) If $x_0 = 1$, then $x_1 = 1 + 1/2 = 3/2 = 1.5$ and $x_2 = 1 + 1/3 = 4/3 = 1.333....$

(b) The iteration can be visualized by the below graph. It is apparent that it converges.

(c) Notice that for $x \geq 1$, the derivative of $g(x)$ satisfies:

$$|g'(x)| = 1/2x^2 \leq 1/2 < 1.$$

This is enough to guarantee that the fixed-point iteration converges.