Math 5031 - Final

Due 12/19/05

In this exam you'll prove a number of famous impossibility results. Here's some background on geometric constructions using a compass and an (unmarked) straightedge, taken from the text by Larry Grove. Choose two points on the plane, which we label as $(0, 0)$ and $(1, 0)$, thus choosing a unit distance and specifying an x-axis. With a straightedge we may draw line segments joining previously constructed points. With a compass we may draw circles having previously constructed centers and radii. We say that a point is *constructible* if it can be obtained as a point of intersection of such lines and/or circles. Recall from high school geometry that a perpendicular to a line can be drawn at a given point on that line. Clearly then any point (a, b), with $a, b \in \mathbb{Z}$, is constructible.

A real number is called *constructible* if it appears as a coordinate for a constructible point in the plane. Denote by K the set of all constructible real numbers. Thus $\mathbb{Z} \subseteq K$.

1. Prove that the set K of constructible numbers is a field and that $a \in K$ if and only if there is a sequence

$$L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots \subseteq L_k$$

   of subfields of $\mathbb{R}$ such that $L_0 = \mathbb{Q}$, $[L_i : L_{i-1}] = 2$ for $1 \leq i \leq k$, and $a \in L_k$. (Note: the key point is that intersection points of lines/circles with lines/circles satisfy quadratic equations.)

2. Is the field K of constructible numbers Galois over $\mathbb{Q}$?

3. **Trisection problem.** We take as a definition that an angle is *constructible* if its cosine and sine are constructible numbers. Show that the angle $\pi/3$, which is constructible, cannot be trisected by ruler and compass. (Hint: let $x = \cos(\pi/9)$ and find a polynomial equation that x must satisfy.)

4. **Regular polygons.** A regular polygon with 18 sides cannot be constructed by compass and straightedge.

5. **Squaring of a circle.** Show that the length of the side of a square having the same area as a given circle of radius 1 is not constructible by compass and straightedge.

6. **Duplicating a cube.** Show that the length of the side of a cube with twice the volume of a given cube (say, of side 1) is not constructible by compass and straightedge.

7. Let p be a prime and $f(X) \in \mathbb{Q}[X]$ an irreducible polynomial over $\mathbb{Q}$ of degree p. If $f(X)$ has exactly two non-real roots, show that the Galois group of $f(X)$ over $\mathbb{Q}$ is isomorphic to the symmetric group S_p.

8. **Unsolvability by radicals.** Show that $f(X) = X^5 - 6X + 3$ is not solvable by radicals.