Math 5031 - Homework 10

Due 11/18/05

1. Let $K = \mathbb{Q}(\alpha)$, where α is a root of the equation

$$\alpha^3 + \alpha^2 + \alpha + 2 = 0.$$

   Express $(\alpha^2+\alpha+1)(\alpha^2+\alpha)$ and $(\alpha-1)^{-1}$ in the form $a\alpha^2+b\alpha+c$, with $a, b, c \in \mathbb{Q}$.

2. Let α be an algebraic element over a field k. Suppose that α has odd degree. Show that $k(\alpha^2) = k(\alpha)$.

3. Let K be a field extension of k. Given algebraic elements $a_1, \ldots, a_n \in K$ over k, show that $k[a_1, \ldots, a_n] = k(a_1, \ldots, a_n)$.

4. Show that $a = 2^{1/3} + 5^{1/4}$ is root of a nonzero polynomial with coefficients in $\mathbb{Q}$ having degree at most 12. (Can you find such a polynomial explicitly? You do not have to!)

5. Show that $\sqrt{2} + \sqrt{3}$ is algebraic over $\mathbb{Q}$ of degree 4.

6. The set $\mathbb{A} \subset \mathbb{C}$ of all algebraic numbers (over $\mathbb{Q}$) is a field extension of $\mathbb{Q}$. (This was shown in class.) Show that $[\mathbb{A} : \mathbb{Q}]$ is infinite.

7. Let E, F be two finite extensions of a field k, contained in a larger field K. Show that

$$[EF : k] \leq [E : k][F : k].$$

   If $[E : k]$ and $[F : k]$ are relatively prime, show that one has equality in the above relation.