Math 5031 - Homework 11

Due 11/28/05

It will be shown in this assignment that the field $\mathbb{C}$ of complex numbers is algebraically closed. We regard $\mathbb{C}$ as the splitting field of the polynomial X^2+1 over $\mathbb{R}$, so that

$$\mathbb{C} = \mathbb{R}(\sqrt{-1}) = \mathbb{R}[X]/(X^2+1).$$

1. Using basic facts from real analysis, show that a real polynomial of odd degree has at least one root in $\mathbb{R}$. Conclude that there are no non-trivial field extensions over $\mathbb{R}$ of odd degree.

2. Show that $\mathbb{C}$ does not have extensions of degree 2. (Note: use that every complex number is the square of a complex number.)

3. Let $p(X) = a_0 + a_1 X + \cdots + a_n X^n \in \mathbb{C}[X]$. Let $\bar{p}(X)$ be the complex conjugate of $p(X)$, that is, $\bar{p}(X) = \bar{a}_0 + \bar{a}_1 X + \cdots + \bar{a}_n X^n$, and define

   $$q(X) = (X^2+1)p(X)\bar{p}(X).$$

   Show that $q(X) \in \mathbb{R}[X]$.

4. Let L denote the splitting field of $q(X)$ over $\mathbb{R}$. Show that L is a Galois extension. Let $G = G(L : \mathbb{R})$ be the Galois group of L over $\mathbb{R}$. Let n be non-negative integer and m an odd positive integer such that $|G| = 2^n m$. Show that $n \geq 1$.

5. Using Sylow's theorem and the result of part 1 above, show that $m = 1$.

6. Using once again Sylow's theorem and part 2 above, show that $n = 1$, hence $L = \mathbb{C}$. Conclude that $\mathbb{C}$ is algebraically closed.