Transcendence of $\pi$ (in ten easy steps). The goal of this assignment is to derive the result, first proved by Lindemann in 1882, that $\pi$ is transcendental over $\mathbb{Q}$. We’ll make use of the following theorem about symmetric polynomials, which I expect to discuss in class either now or in the Spring. A polynomial $f(X_1, \ldots, X_n) \in R[X_1, \ldots, X_n]$, with coefficients in a ring $R$, is said to be a symmetric polynomial over $R$ if

$$f(X_1, \ldots, X_n) = f(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$$

for all permutations $\sigma \in S_n$.

**Theorem 1** Let $f(X) \in \mathbb{Z}[X]$ be a polynomial with leading coefficient $b$ and let $\alpha_1, \ldots, \alpha_n$ be the roots of $f(X)$ in $\mathbb{C}$. If $g(X_1, \ldots, X_n)$ is any symmetric polynomial over $\mathbb{Q}$ of degree $d$, then the following hold:

1. $g(\alpha_1, \ldots, \alpha_n) \in \mathbb{Q}$;
2. if $g(X_1, \ldots, X_n)$ has coefficients in $\mathbb{Z}$, then $b^d g(a_1, \ldots, a_n) \in \mathbb{Z}$.

We now begin the proof of Lindemann’s theorem. Arguing by contradiction, suppose that $p(X) = a_0 + a_1 X + \cdots + a_n X^n$ is a non-zero polynomial in $\mathbb{Z}[X]$ having $i\pi$ as a root. (If $i\pi$ is shown to be transcendental over $\mathbb{Q}$, then clearly so must be $\pi$.)

1. Set $a = a_n$ and $b_k = a_n^{-1-k}a_k$, for $k = 0, \ldots, n-1$. Check that $q(X) = b_0 + b_1 X + \cdots + b_{n-1} X^{n-1} + X^n$ has roots $a\alpha_1, \ldots, a\alpha_n$, where $\alpha_k$ are the roots of $p(X)$. In particular, $i\pi a$ is a root of $q(X)$.

2. Show that

$$0 = \prod_{k=1}^{n} (e^{i\alpha_k} + 1) = \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} e^{k_1\alpha_1 + \cdots + k_n\alpha_n}.$$

3. Consider the polynomial

$$u(X) = \prod_{k_1=0}^{1} \prod_{k_n=0}^{1} \left( X - a \sum_{i=1}^{n} k_i \alpha_i \right).$$
Observe that the roots of \( u(X) \) are

\[ a(\alpha_{k_1} + \cdots + \alpha_{k_i}), \quad 1 \leq i \leq n, \quad 1 \leq k_1 < k_2 < \cdots < k_i \leq n, \]

and that

\[ u(X) = X^{2^n} + A_{2^n-1}(\alpha_1, \ldots, \alpha_n)X^{2^{n-1}} + \cdots + A_0(\alpha_1, \ldots, \alpha_n), \]

where the coefficients \( A_j \) lie in \( \mathbb{Z}[\alpha_1, \ldots, \alpha_n] \). Show that the \( A_j \) are symmetric polynomials in the \( \alpha_i \). Conclude that \( u(X) \in \mathbb{Z}[X] \).

4. Let \( \beta_1, \ldots, \beta_r \) be complex numbers such that \( a\beta_i \) are the non-zero roots of \( u(X) \). Show that

\[ \sum_{j=1}^{r} e^{\beta_j} = -l, \]

where \( l \) is the multiplicity of the root 0. Denote by \( s(X) \) the monic polynomial of degree \( r \) obtained by dividing \( u(X) \) by \( X^l \).

5. Now define for each prime \( p \) the polynomials

\[ f_p(X) = \frac{(aX)^{p-1}(s(aX))^p}{(p-1)!}, \quad F_p(X) = \sum_{k=0}^{p-1} f_p^{(k)}(X), \]

where \( f_p^{(k)}(X) \) denotes the \( k \)th derivative of \( f_p(X) \). Prove the following:

\[ e^{-\beta}F_p(\beta) - F_p(0) = -\int_0^1 \beta e^{-\beta x} f_p(\beta x)dx \quad (1) \]
\[ \sum_{k=1}^{r} F_p(\beta_k) + lF_p(0) = -\sum_{k=1}^{r} \beta_k \int_0^1 e^{(1-x)\beta_k} f_p(\beta_k x)dx. \quad (2) \]

6. For each prime \( p \) and each \( \beta_k \), define:

\[ T_p(\beta_k) = \beta_k \int_0^1 e^{(1-x)\beta_k} f_p(\beta_k x)dx. \]

Show that

\[ |T_p(\beta_k)| \leq K_kH_k \left( |a\beta_k|H_k \right)^{p-1}, \]

where

\[ H_k = \sup_{0 \leq x \leq 1} |s(a\beta_k x)| \text{ and } K_k = |\beta_k e^{\beta_k}| \int_0^1 |e^{-\beta_k x}| dx. \]
7. Show that for sufficiently large $p$, we have

$$\left| \sum_{k=1}^{r} T_p(\beta_k) \right| \leq \frac{1}{2}.$$ 

Therefore,

$$\left| \sum_{k=1}^{r} F_p(\beta_k) + lF_p(0) \right| \leq \frac{1}{2}.$$ 

We will derive a contradiction by also showing in the remaining steps that

$$\left| \sum_{k=1}^{r} F_p(\beta_k) + lF_p(0) \right| \geq 1.$$ 

8. Using the Leibniz formula (for the $h$th derivative of a product), express $f_p^{(h)}(X)$ in terms of the derivatives of $s^p(aX)$ and show the following:

(a) If $0 \leq h < p$, then $\sum_{k=1}^{r} f_p^{(h)}(\beta_k) = 0,$
(b) If $h \geq p$, then $\sum_{k=1}^{r} f_p^{(h)}(\beta_k)$ is an integer divisible by $p$.
(c) Conclude that $\sum_{k=1}^{r} F_p(\beta_k)$ is an integer divisible by $p$.

(Hint for part (b): write $(p-1)!f_p(X) = \sum_{j \geq 0} c_j X^j$, where the $c_j$ are integers, and note that

$$\sum_{k=1}^{n} \left( \frac{d^h}{dX^h} \sum_{j \geq 0} c_j X^j \right) \bigg|_{X=\beta_k}$$

is integer by another application of the above theorem on symmetric functions.)

9. By studying $f_p^{(j)}(0)$, show that there exists an integer $M$ such that

$$\sum_{k=1}^{r} F_p(\beta_k) + lF_p(0) = Mp + l\alpha^{p-1}s^p(0).$$

From this, show that for all prime $p$ strictly greater than the maximum among $l, |a|, |s(0)|$, we have

$$\sum_{k=1}^{r} F_p(\beta_k) + lF_p(0) \neq 0.$$ 

10. Derive a contradiction, thus proving that $i\pi$ is transcendental.

One more step. You may wish to go one extra step now and look up in your favorite algebra text the proof of the theorem on symmetric functions. As I mentioned above, I hope to discuss it in class before long. But, having done all this work, you will probably want to convince yourself that what is left is a much easier fact than what we have just proved.