Math 5031 - Homework 13

Due 12/12/05

Transcendence of π (in ten easy steps). The goal of this assignment is to derive the result, first proved by Lindemann in 1882, that π is transcendental over $\mathbb{Q}$. We'll make use of the following theorem about symmetric polynomials, which I expect to discuss in class either now or in the Spring. A polynomial $f(X_1, \ldots, X_n) \in R[X_1, \ldots, X_n]$, with coefficients in a ring R, is said to be a *symmetric polynomial* over R if

$$f(X_1, \ldots, X_n) = f(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$$

for all permutations $\sigma \in S_n$.

Theorem 1 *Let $f(X) \in \mathbb{Z}[X]$ be a polynomial with leading coefficient b and let $\alpha_1, \ldots, \alpha_n$ be the roots of $f(X)$ in $\mathbb{C}$. If $g(X_1, \ldots, X_n)$ is any symmetric polynomial over $\mathbb{Q}$ of degree d, then the following hold:*

1. *$g(\alpha_1, \ldots, \alpha_n) \in \mathbb{Q}$;*
2. *if $g(X_1, \ldots, X_n)$ has coefficients in $\mathbb{Z}$, then $b^d g(a_1, \ldots, a_n) \in \mathbb{Z}$.*

We now begin the proof of Lindemann's theorem. Arguing by contradiction, suppose that $p(X) = a_0 + a_1 X + \cdots + a_n X^n$ is a non-zero polynomial in $\mathbb{Z}[X]$ having $i\pi$ as a root. (If $i\pi$ is shown to be transcendental over $\mathbb{Q}$, then clearly so must be π.)

1. Set $a = a_n$ and $b_k = a^{n-1-k} a_k$, for $k = 0, \ldots, n-1$. Check that $q(X) = b_0 + b_1 X + \cdots + b_{n-1} X^{n-1} + X^n$ has roots $a\alpha_1, \ldots, a\alpha_n$, where α_k are the roots of $p(X)$. In particular, $i\pi a$ is a root of $q(X)$.

2. Show that

$$0 = \prod_{k=1}^{n} (e^{\alpha_k} + 1) = \sum_{k_1=0}^{1} \cdots \sum_{k_n=0}^{1} e^{k_1\alpha_1 + \cdots + k_n\alpha_n}.$$

3. Consider the polynomial

$$u(X) = \prod_{k_1=0}^{1} \cdots \prod_{k_n=0}^{1} \left(X - a \sum_{i=1}^{n} k_i \alpha_i \right).$$

Observe that the roots of $u(X)$ are

$$a(\alpha_{k_1} + \cdots + \alpha_{k_i}), \quad 1 \leq i \leq n, \ 1 \leq k_1 < k_2 < \cdots < k_i \leq n,$$

and that

$$u(X) = X^{2^n} + A_{2^n-1}(\alpha_1, \ldots, \alpha_n)X^{2^n-1} + \cdots + A_0(\alpha_1, \ldots, \alpha_n),$$

where the coefficients A_j lie in $\mathbb{Z}[\alpha_1, \ldots, \alpha_n]$. Show that the A_j are symmetric polynomials in the α_i. Conclude that $u(X) \in \mathbb{Z}[X]$.

4. Let $\beta_1, \ldots, \beta_r$ be complex numbers such that $a\beta_i$ are the non-zero roots of $u(X)$. Show that

$$\sum_{j=1}^{r} e^{\beta_j} = -l,$$

where l is the multiplicity of the root 0. Denote by $s(X)$ the monic polynomial of degree r obtained by dividing $u(X)$ by X^l.

5. Now define for each prime p the polynomials

$$f_p(X) = \frac{(aX)^{p-1}(s(aX))^p}{(p-1)!}, \quad F_p(X) = \sum_{k=0}^{pr+p-1} f_p^{(k)}(X),$$

where $f_p^{(k)}(X)$ denotes the kth derivative of $f_p(X)$. Prove the following:

$$e^{-\beta}F_p(\beta) - F_p(0) = -\int_0^1 \beta e^{-\beta x} f_p(\beta x)dx \tag{1}$$

$$\sum_{k=1}^{r} F_p(\beta_k) + lF_p(0) = -\sum_{k=1}^{r} \beta_k \int_0^1 e^{(1-x)\beta_k} f_p(\beta_k x)dx. \tag{2}$$

6. For each prime p and each β_k, define:

$$T_p(\beta_k) = \beta_k \int_0^1 e^{(1-x)\beta_k} f_p(\beta_k x)dx.$$

Show that

$$|T_p(\beta_k)| \leq K_k H_k \frac{(|a\beta_k|H_k)^{p-1}}{(p-1)!},$$

where

$$H_k = \sup_{0 \leq x \leq 1} |s(a\beta_k x)| \text{ and } K_k = |\beta_k e^{\beta_k}| \int_0^1 |e^{-\beta_k x}|dx.$$

7. Show that for sufficiently large p, we have

$$\left|\sum_{k=1}^{r} T_p(\beta_k)\right| \leq \frac{1}{2}.$$

   Therefore,

$$\left|\sum_{k=1}^{r} F_p(\beta_k) + lF_p(0)\right| \leq \frac{1}{2}.$$

   We will derive a contradiction by also showing in the remaining steps that

$$\left|\sum_{k=1}^{r} F_p(\beta_k) + lF_p(0)\right| \geq 1.$$

8. Using the Leibniz formula (for the hth derivative of a product), express $f_p^{(h)}(X)$ in terms of the derivatives of $s^p(aX)$ and show the following:

   (a) If $0 \leq h < p$, then $\sum_{k=1}^{r} f_p^{(h)}(\beta_k) = 0$,

   (b) If $h \geq p$, then $\sum_{k=1}^{r} f_p^{(h)}(\beta_k)$ is an integer divisible by p.

   (c) Conclude that $\sum_{k=1}^{r} F_p(\beta_k)$ is an integer divisible by p.

   (Hint for part (b): write $(p-1)!f_p(X) = \sum_{j\geq 0} c_j X^j$, where the c_j are integers, and note that

$$\sum_{k=1}^{n} \left(\frac{d^h}{dX^h} \sum_{j\geq 0} c_j X^j \right)_{X=\beta_k}$$

   is integer by another application of the above theorem on symmetric functions.)

9. By studying $f_p^{(j)}(0)$, show that there exists an integer M such that

$$\sum_{k=1}^{r} F_p(\beta_k) + lF_p(0) = Mp + la^{p-1}s^p(0).$$

   From this, show that for all prime p strictly greater than the maximum among $l, |a|, |s(0)|$, we have

$$\sum_{k=1}^{r} F_p(\beta_k) + lF_p(0) \neq 0.$$

10. Derive a contradiction, thus proving that $i\pi$ is transcendental.

One more step. You may wish to go one extra step now and look up in your favorite algebra text the proof of the theorem on symmetric functions. As I mentioned above, I hope to discuss it in class before long. But, having done all this work, you will probably want to convince yourself that what is left is a much easier fact than what we have just proved.