Math 5031 - Homework 2
Due 9/16/05

1. **Semidirect products.** We define $G$ to be a semidirect product of subgroups $H$ and $N$ if $N$ is normal, $G = NH$ and $H \cap N = \{e\}$.

   (a) Let $G$ be a group and let $H$, $N$ be subgroups with $N$ normal. Let $\gamma_x$ be conjugation by an element $x \in G$. Show that $x \rightarrow \gamma_x$ induces a homomorphism $f : H \rightarrow \text{Aut}(N)$.

   (b) If $H \cap N = \{e\}$, show that the map $H \times N \rightarrow HN$ given by $(x, y) \mapsto xy$ is a bijection, and that this map is an isomorphism if and only if $f$ is trivial, i.e., $f(x) = \text{id}_N$ for all $x \in H$.

   (c) Conversely, let $N$ and $H$ be groups, and let $\psi : H \rightarrow \text{Aut}(N)$ be a given homomorphism. Construct a semidirect product as follows. Let $G$ be the set of pairs $(x, h) \in N \times H$. Define the composition law

   $$(x_1, h_1)(x_2, h_2) = (x_1\psi_1(h_1)x_2, h_1h_2).$$

   Show that this is a group law, and yields a semidirect product of $N$ and $H$, identifying $N$ with the set of elements $(x, 1)$, and $H$ with the set of elements $(1, h)$.

   (d) Suppose that $N$ and $H$ are both normal subgroups of $G$ and that the orders of $N$ and $H$ are relatively prime. Prove that $HN$ a subgroup of $G$ isomorphic to the direct product $H \times N$.

   (e) Let $G$ be a finite group and let $N$ be a normal subgroup such that $N$ and $G/N$ have relatively prime orders. Let $H$ be a subgroup of $G$ having the same order as $G/H$. Show that $G$ is the semidirect product of $N$ and $H$. Also show that if $\sigma$ is any automorphism of $G$, then $\sigma(N) = N$.

2. **Group actions.** We say that a group action is transitive if it has a single orbit.

   (a) Show that a transitive action of a group $G$ on a set $X$ is equivalent to the action of $G$ on the right-coset space $G/H$ by left-translations.

   (b) Let $G$ be a group acting transitively on a finite set $X$, where $\#X \geq 2$. Prove that there exists an element $g$ of $G$ which has no fixed point, i.e., $gx \neq x$ for all $x \in X$. (Hint: first prove the next item.)

   (c) Let $H$ be a proper subgroup of a finite group $G$. Show that $G$ is not the union of all the conjugates of $H$. 

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