# Math 5031 - Homework 5

Due 10/07/05

1. An abelian group $A$ is said to be a *torsion group* if every element of $A$ has finite order. Show that the additive group $\mathbb{Q}/\mathbb{Z}$ is torsion, and that it has one and only one subgroup of order $n$, for each positive integer $n$. Furthermore, show that this subgroup of order $n$ is cyclic.

2. Describe all abelian subgroups of order $n$, up to isomorphism, for $n = 24, 200, 1000, p^3$, and $p^4$.

3. We denote by $Z(H)$ the center of a group $H$. The *ascending central series* of a group $G$ is, by definition, the sequence
$$\{e\} = Z_0 \subset Z_1 = Z(G) \subset Z_2 \subset Z_3 \subset \dots$$
where $Z_{i+1}/Z_i = Z(G/Z_i)$, for all $i$. We say that $G$ is *nilpotent* if $Z_n = G$ for some positive integer $n$. Show that if $G = G_1 G_2 \dots G_k$ is a direct product where each $G_i$ is a $p_i$-Sylow subgroup of $G$ for distinct prime numbers $p_i$, then $G$ is nilpotent. (It turns out that the converse is also true: A nilpotent finite group $G$ is the direct product of its Sylow $p$-groups. Can you prove this fact? You do not need to write it down.)

4. Define the *descending central series* of a group $G$ by setting $L_0 = G$, $L_1 = [G, G]$, and $L_{k+1} = [G, L_k]$ for all $k \geq 1$. It is not difficult to see (you do not need to show it here) that each $L_k$ is normal in $G$, so that $L_{k+1} \subset L_k$ for each $k$. Show that a group $G$ is nilpotent if and only if $L_n(G) = \{e\}$ for some positive integer $n$.

5. Let $GL(n, \mathbb{R})$ denote the group of all invertible real valued $n \times n$ matrices and let $B$ be the subset of all upper-triangular matrices with diagonal entries equal to 1. Show that $B$ is a nilpotent group. (If $B'$ is the subgroup of upper-triangular matrices, with arbitrary elements on the diagonal, then $B$ is solvable, but not nilpotent. You don't have to write down the proof of this claim.)