Math 5031 - Homework 6

Due 10/14/05

1. Define the *generalized quaternion group* Q_m by the presentation

$$Q_m = \langle a, b \mid a^{2m} = 1, b^2 = a^m, ab = ba^{-1} \rangle$$

   for $m \geq 1$. Show that Q_m has order $4m$. Find a concrete group with order $4m$ having the same relations as Q_m. (Hint: consider $e^{i\pi/m}$ and j inside the ring of quaternions.)

2. Use generators $a = (12), b = (23)$, and $c = (34)$ for S_4 and show that S_4 has the presentation

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (ac)^2 = (bc)^3 = 1 \rangle.$$

3. Let R be the set of functions $f : \mathbb{N} \to K$, where K is a commutative ring with 1. Define the sum in R to be the ordinary addition of functions, and define the *convolution product* by the formula

$$(f * g)(m) = \sum_{xy=m} f(x)g(y),$$

   where the sum is taken over all pairs (x, y) of positive integers such that $xy = m$.

   (a) Show that R is a commutative ring, whose unit element is the function δ such that $\delta(1) = 1$ and $\delta(x) = 0$ if $x \neq 1$.

   (b) A function is said to be *multiplicative* if $f(mn) = f(m)f(n)$ whenever m and n are relatively prime. If f, g are multiplicative, show that $f * g$ is multiplicative.

   (c) Let m be the *Möbius function* such that $\mu(1) = 1$, $\mu(p_1 \ldots p_r) = (-1)^r$ if $p_1, \ldots, p_r$ are distinct primes, and $\mu(m) = 0$ if m is divisible by p^2 for some prime p. Show that $\mu * \phi_1 = \delta$, where ϕ_1 denotes the constant function having value 1.