Math 5031 - Homework 6

Due 10/14/05

1. Define the generalized quaternion group $Q_m$ by the presentation

$$Q_m = \langle a, b \mid a^{2m} = 1, b^2 = a^m, ab = ba^{-1} \rangle$$

for $m \geq 1$. Show that $Q_m$ has order $4m$. Find a concrete group with order $4m$ having the same relations as $Q_m$. (Hint: consider $e^{i\pi/m}$ and $j$ inside the ring of quaternions.)

2. Use generators $a = (12), b = (23), \text{ and } c = (34)$ for $S_4$ and show that $S_4$ has the presentation

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (ac)^2 = (bc)^3 = 1 \rangle.$$ 

3. Let $R$ be the set of functions $f : \mathbb{N} \to K$, where $K$ is a commutative ring with 1. Define the sum in $R$ to be the ordinary addition of functions, and define the convolution product by the formula

$$(f \ast g)(m) = \sum_{xy = m} f(x)g(y),$$

where the sum is taken over all pairs $(x, y)$ of positive integers such that $xy = m$.

(a) Show that $R$ is a commutative ring, whose unit element is the function $\delta$ such that $\delta(1) = 1$ and $\delta(x) = 0$ if $x \neq 1$.

(b) A function is said to be multiplicative if $f(mn) = f(m)f(n)$ whenever $m$ and $n$ are relatively prime. If $f, g$ are multiplicative, show that $f \ast g$ is multiplicative.

(c) Let $m$ be the Möbius function such that $\mu(1) = 1$, $\mu(p_1 \ldots p_r) = (-1)^r$ if $p_1, \ldots, p_r$ are distinct primes, and $\mu(m) = 0$ if $m$ is divisible by $p^2$ for some prime $p$. Show that $\mu \ast \phi_1 = \delta$, where $\phi_1$ denotes the constant function having value 1.