Math 5031 - Homework 8

Due 11/04/05

1. Give an example of a ring R with a prime ideal $P \neq 0$ that is not maximal.

2. Show that $\mathbb{Z}[\sqrt{-5}]$ is not a P.I.D. (Hint: Consider the elements $3, 2-\sqrt{-5}$, $2+\sqrt{-5}$. Show that 3 is irreducible but not prime.)

3. Show that the ideal $I = (2, x)$ is not principal in $\mathbb{Z}[x]$.

4. Let S be an integral domain with 1 and set $R = S[x, y]$. Show that R is not an Euclidean domain. (Hint: Show that the ideal $I = (x, y)$ is not principal.)

5. If R is the subring of $\mathbb{Z}[x]$ consisting of all
$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$
such that a_k is even for $1 \leq k \leq n$, show that R is not Noetherian. (Hint: Consider the ideal generated by $\{2x^k : 1 \leq k \in \mathbb{Z}\}$.)

6. Suppose $f(x) = 1 + x + x^2 + \cdots + x^{p-1}$, where p is a prime in $\mathbb{Z}$. Show that $f(x)$ is irreducible in $\mathbb{Q}[x]$. (Hint: Write $f(x) = (x^p - 1)/(x - 1)$; substitute $x + 1$ for x.)

7. If $p \in \mathbb{Z}$ is prime and $1 < m \in \mathbb{Z}$, show that $p^{1/m}$ is irrational. (Hint: show first that $x^m - p$ is irreducible in $\mathbb{Q}[x]$.)

8. Solve the following systems of congruence equations. (The rings are, respectively, $R = \mathbb{Z}$, $\mathbb{R}[i]$, and $F[x]$, where F is a field in which $1 + 1 \neq 0$.)

   (a) $x \equiv 1 (\text{mod } 8), \quad x \equiv 3 (\text{mod } 7), \quad x \equiv 9 (\text{mod } 11)$.

   (b) $x \equiv i (\text{mod } 1 + i), \quad x \equiv 1 (\text{mod } 2 - i), \quad x \equiv 1 + i (\text{mod } 3 + 4i)$.

   (c) $f(x) \equiv 1 (\text{mod } x-1), \quad f(x) \equiv x (\text{mod } x^2+1), \quad f(x) \equiv x^3 (\text{mod } x+1)$.