

# Math 5032 - Homework 1

Due 1/27/06

1. Give an example of an  $R$ -module  $M$  having  $R$ -isomorphic submodules  $N_1$  and  $N_2$  such that  $M/N_1$  and  $M/N_2$  are not isomorphic.
2. Suppose  $V$  is a finite-dimensional vector space over a field  $k$ , viewed as a  $k$ -module. Describe a composition series for  $V$  and determine its factors.
3. A sequence of  $R$ -homomorphisms of  $R$ -modules  $K \xrightarrow{f} M \xrightarrow{g} N$  is *exact* at  $M$  if  $\text{Im}(f) = \ker(g)$ . A *short exact sequence*  $0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0$  is exact at  $K$ ,  $M$ , and  $N$ . If

$$0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0$$

is a short exact sequence show that  $M$  is Noetherian if and only if  $K$  and  $N$  are both Noetherian.

4. Suppose  $M_1$ ,  $M_2$  and  $N$  are submodules of an  $R$ -module  $M$ , with  $M_1 \subseteq M_2$ . Show that there is an exact sequence

$$0 \rightarrow (M_2 \cap N)/(M_1 \cap N) \rightarrow M_2/M_1 \rightarrow (M_2 + N)/(M_1 + N) \rightarrow 0.$$

5. Suppose  $L$ ,  $M$ , and  $N$  are  $R$ -modules and  $f : M \rightarrow N$  is an  $R$ -homomorphism. Define

$$f^* : \text{Hom}_R(N, L) \rightarrow \text{Hom}_R(M, L)$$

via  $f^*(\phi) : m \mapsto \phi(f(m))$  for all  $\phi \in \text{Hom}_R(N, L)$  and  $m \in M$ .

- (a) Show that  $f^*$  is a  $\mathbb{Z}$ -homomorphism.
- (b) If  $R$  is commutative show that  $f^*$  is an  $R$ -homomorphism.

- (c) If  $K$ ,  $M$ , and  $N$  are  $R$ -modules, show that  $K \rightarrow M \rightarrow N \rightarrow 0$  is an exact sequence of  $R$ -homomorphisms if and only if

$$0 \rightarrow \text{Hom}_R(N, L) \rightarrow \text{Hom}_R(M, L) \rightarrow \text{Hom}_R(K, L)$$

is an exact sequence of  $\mathbb{Z}$ -homomorphisms for all  $R$ -modules  $L$ .

6. Suppose  $R$  is a commutative ring with 1 and  $M$  is an  $R$ -module. Then the  $R$ -module  $M^* = \text{Hom}_R(M, R)$  is called the *dual module* of  $M$ . The elements of  $M^*$  are commonly called  *$R$ -linear functionals* on  $M$ . If  $M$  is free of finite rank, with basis  $\{x_1, x_2, \dots, x_n\}$ , show that  $M^*$  is also free, with basis  $\{f_1, f_2, \dots, f_n\}$ , where

$$f_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

(The *dual basis*.) Conclude that  $M$  and  $M^*$  are  $R$ -isomorphic in that case.