Math 5032 - Homework 10

Due 4/24/06

1. Suppose a finite group G acts on a finite set S by permutations and let T denote the permuation representation of G. (H.W. 9, problem 2.) Let χ be the character of T.

   (a) If G is transitive on S, show that $T = 1 \oplus L$, where L does not contain the trivial representation 1. If θ denote the character of L, it follows that $\chi = 1 + \theta$ and $(\theta, 1) = 0$.

   (b) Let G act on the product $S \times S$ according to $\sigma(x, y) = (\sigma x, \sigma y)$. Show that the character of the corresponding permutation representation is χ^2.

   (c) Suppose that G is transitive on S and S has at least 2 elements. We say that G is *doubly transitive* if for all x, y, x', y' in S such that $x \neq y$, $x' \neq y'$, there exists $\sigma \in G$ such that $x' = \sigma x$ and $y' = \sigma y$. Prove that the following are equivalent:

      i. G is doubly transitive on S.

      ii. The action of G on $S \times S$ has exactly two orbits: the diagonal and its complement.

      iii. $(\chi^2, 1) = 2$

      iv. The permutation representation L define in part 1 for the action of G on S is irreducible.

      Therefore, G is doubly transitive on S if and only if the permutation representation splits as a direct sum of the trivial representation and an irreducible representation.

2. Let T be an irreducible representation of G of degree l and character χ. Let C be the center of G and $c = |C|$, $n = |G|$.

(a) Show that $|\chi(\sigma)| = l$. (First check that $T(\sigma) = zI$, for some $z \in \mathbb{C}$. Obtain the value of z and take determinants.)

(b) Show that $l^2 \leq n/c$.

(c) Show that if T is a faithful representation, the group C is cyclic. (Obtain a faithful representation of C in $\mathbb{C}$ and use that a finite subgroup of $\mathbb{C}$ is cyclic.)

3. Let G be an abelian group of order n, and let $\hat{G}$ be the set of irreducible characters of G. If χ_1, χ_2 belong to $\hat{G}$, then $\chi_1\chi_2$ also belongs to $\hat{G}$. Show that this makes $\hat{G}$ an abelian group of order n. The group $\hat{G}$ is called the *dual group* of G. For $\sigma \in G$ the mapping $\chi \mapsto \chi(\sigma)$ is a irreducible character of $\hat{G}$ and so an element of the dual $\hat{\hat{G}}$ of $\hat{G}$. Show that the map of G into $\hat{\hat{G}}$ thus obtained is an injective homomorphism. Conclude that it is an isomorphism.

4. Find the character table of the dihedral group D_6 of symmetries of the regular hexagon.