Math 5032 - Homework 3

Due 2/17/06

1. Suppose R is a ring with 1. A unitary R-module P is called *projective* if given an exact sequence $M \xrightarrow{g} N \to 0$ of R-modules and an R-homomorphism $f : P \to N$, then there is an R-homomorphism $h : P \to M$ such that $f = g \circ h$.

   (a) Show that free modules are projective.

   (b) If $P = P_1 \oplus P_2$, show that P is projective if and only if both P_1 and P_2 are projective. Generalize to arbitrary direct sums.

2. Show that an R-module P is projective if and only if P is a direct summand of some free module F.

3. If R is a PID or a division ring show that every projective R-module is free.

4. Show the following, where $m, p, q \in \mathbb{Z}$ and p, q are distinct primes:

   (a) $\mathbb{Z}/(m)$ is not a projective $\mathbb{Z}$-module.

   (b) $\mathbb{Z}/(p)$ is a projective $\mathbb{Z}/(pq)$-module but not a free $\mathbb{Z}/(pq)$-module.

   Give an example of a projective R-module that is not free.

5. Show that an R-module P is projective if and only if given any short exact sequence $0 \to K \to M \to P \to 0$ we may conclude that $M \cong K \oplus P$. (We say that the sequence *splits*.)

6. Let R be a ring with 1. A unitary R-module Q is called *injective* if given any exact sequence $0 \to K \xrightarrow{g} M$ of R-modules and a homomorphism $f : K \to Q$, then there is a homomorphism $h : M \to Q$ such that $f = h \circ g$. If $Q = Q_1 \times Q_2$ show that Q is injective if and only if both Q_1 and Q_2 are injective. Generalize to arbitrary direct products.

7. Show that if an R-module Q is injective, then given any short exact sequence $0 \to Q \to M \to N \to 0$ of R-modules we may conclude that $M \cong Q \oplus N$ (the sequence splits) or, equivalently, Q is a direct summand of every R-module that contains it as a submodule.