Math 5032 - Homework 5

Due 3/03/06

1. If R is a commutative ring with 1 and x_1, x_2 are distinct indeterminates show that $R[x_1, x_2]$ and $R[x_1] \otimes R[x_2]$ are isomorphic as R-algebras.

2. If k is a field and K is an extension field of k show that $M_n(K) \cong K \otimes_k M_n(k)$ as k-algebras.

3. If A is an abelian group, show that $\mathbb{Z}/(n) \otimes_{\mathbb{Z}} A \cong A/nA$.

4. If A and B are finitely generated abelian groups show that $A \otimes_{\mathbb{Z}} B$ is a finitely generated abelian group. Find its rank and elementary divisors in terms of those of A and B.

5. Suppose R is a commutative ring with 1 and E, F are free R-modules, with respective basis $\{x_\alpha : \alpha \in A\}$ and $\{y_\beta : \beta \in B\}$. Show that $E \otimes_R F$ is free with basis $\{x_\alpha \otimes y_\beta : \alpha \in A, \beta \in B\}$.

6. Suppose R is a commutative ring and L, M, N are R-modules. Show that $\mathrm{Hom}_R(L, \mathrm{Hom}_R(M, N))$ and $\mathrm{Hom}_R(L \otimes_R M, N)$ are isomorphic R-modules. Conclude in particular that $(L \otimes_R M)^* \cong \mathrm{Hom}_R(L, M^*)$.

7. If $K \to M \to N \to 0$ is an exact sequence of (left) R-modules and L is a right R-module show that $L \otimes_R K \to L \otimes_R M \to L \otimes_R N \to 0$ is an exact sequence of abelian groups.