In this short assignment you will prove the following simple but fundamental fact about group algebras due to Maschke.

**Theorem 0.1** Let $G$ be a finite group, $F$ a field and $R = FG$ the group algebra of $G$. Let $M$ be a unitary $R$-module having finite dimension as a vector space over $F$. Suppose that the characteristic of $F$ is either 0 or a prime that does not divide $|G|$. Let $M$ be a unitary $R$-module having finite dimension as a vector space over $F$. Then $M$ is a semisimple module.

Give a proof of Maschke’s theorem by following these steps: Let $N$ be a nonzero $R$-submodule of $M$ and choose a vector subspace $V \subset M$ over $F$ such that $M = N \oplus V$ as a vector space over $F$. Let $\pi : M \to N$ denote the standard linear projection along $V$. Define an $F$-linear map $g : M \to M$ by

$$g(m) = |G|^{-1} \sum_{x \in G} x\pi(x^{-1}m).$$

1. Show that $g \in C_R(M)$ (the center of $M$.)
2. Show that $g(n) = n$ for all $n \in N$.
3. Show that $g^2 = g$.
4. Set $K = (I - g)M$. Show that $K$ is an $R$-module such that $M = N \oplus K$.
5. Conclude that $M$ is semisimple.