Math 5032 - Homework 6

Due 3/10/06

In this short assignment you will prove the following simple but fundamental fact about group algebras due to Maschke.

Theorem 0.1 *Let G be a finite group, F a field and $R = FG$ the group algebra of G. Let M be a unitary R-module having finite dimension as a vector space over F. Suppose that the characteristic of F is either 0 or a prime that does not divide $|G|$. Let M be a unitary R-module having finite dimension as a vector space over F. Then M is a semisimple module.*

Give a proof of Maschke's theorem by following these steps: Let N be a nonzero R-submodule of M and choose a vector subspace $V \subset M$ over F such that $M = N \oplus V$ as a vector space over F. Let $\pi : M \to N$ denote the standard linear projection along V. Define an F-linear map $g : M \to M$ by

$$g(m) = |G|^{-1} \sum_{x \in G} x\pi(x^{-1}m).$$

1. Show that $g \in C_R(M)$ (the center of M.)
2. Show that $g(n) = n$ for all $n \in N$.
3. Show that $g^2 = g$.
4. Set $K = (I - g)M$. Show that K is an R-module such that $M = N \oplus K$.
5. Conclude that M is semisimple.