

# Math 5032 - Homework 6

Due 3/10/06

In this short assignment you will prove the following simple but fundamental fact about group algebras due to Maschke.

**Theorem 0.1** *Let  $G$  be a finite group,  $F$  a field and  $R = FG$  the group algebra of  $G$ . Let  $M$  be a unitary  $R$ -module having finite dimension as a vector space over  $F$ . Suppose that the characteristic of  $F$  is either 0 or a prime that does not divide  $|G|$ . Let  $N$  be a unitary  $R$ -submodule having finite dimension as a vector space over  $F$ . Then  $N$  is a semisimple module.*

Give a proof of Maschke's theorem by following these steps: Let  $N$  be a nonzero  $R$ -submodule of  $M$  and choose a vector subspace  $V \subset M$  over  $F$  such that  $M = N \oplus V$  as a vector space over  $F$ . Let  $\pi : M \rightarrow N$  denote the standard linear projection along  $V$ . Define an  $F$ -linear map  $g : M \rightarrow M$  by

$$g(m) = |G|^{-1} \sum_{x \in G} x \pi(x^{-1}m).$$

1. Show that  $g \in C_R(M)$  (the center of  $M$ .)
2. Show that  $g(n) = n$  for all  $n \in N$ .
3. Show that  $g^2 = g$ .
4. Set  $K = (I - g)M$ . Show that  $K$  is an  $R$ -module such that  $M = N \oplus K$ .
5. Conclude that  $M$  is semisimple.