Math 5032 - Homework 7

Due 3/27/06

1. **Nakayama's lemma**. let R be any ring and M a finitely generated unitary R-module. Let $N = J(R)$ be the radical of R. If $NM = M$ show that $M = 0$.

   (Hint: Write $M = R\langle x_1, \ldots, x_s \rangle$ and $x_s = a_1x_1 + \cdots + a_sx_s$, for $a_i \in N$. Observe that $1 - a_s$ must be a unit: if not, consider a maximal ideal containing $1-a_s$ and obtain a contradiction by showing that 1 is in the ideal. Conclude that $x_s \in R\langle x_1, \ldots, x_{s-1} \rangle$, and the claim by induction.)

2. Let J be a two-sided nilpotent ideal of R. Show that J is contained in the Jacobson radical.

3. Suppose that R is Artinian. Show that $N = J(R)$ is a nilpotent ideal.

   (Hint: Let N^∞ be the ideal to which the chain $N \supset N^2 \supset \ldots$ terminates. Suppose that N^∞ is non-zero and let I be a minimal left ideal in N^∞ such NI is non-zero. Argue that I finitely generated and obtain a contradiction using Nakayama's lemma.)

4. Let F be a field and $R = M_2(F)$, the ring of 2×2 matrices over F. Exhibit a left ideal in R that is not a semisimple ring.

5. Suppose F is a field and $f(x) \in F[x]$. Set $R = F[x]/(f(x))$. Determine $J(R)$. Under what condition is R semisimple? (Recall the example $\mathbb{Z}/(n)$ discussed in class.)

6. If F is a field, let R be the ring of $n \times n$ upper triangular matrices with entries from F. (That is, $A = (a_{ij}) \in R$, then $a_{ij} = 0$ if $i > j$.) Find $J(R)$ and show that $R/J(R)$ is commutative.