1. **Nakayama’s lemma.** Let $R$ be any ring and $M$ a finitely generated unitary $R$-module. Let $N = J(R)$ be the radical of $R$. If $NM = M$ show that $M = 0$.

(Hint: Write $M = R\langle x_1, \ldots, x_s \rangle$ and $x_s = a_1x_1 + \cdots + a_sx_s$, for $a_i \in N$. Observe that $1 - a_s$ must be a unit: if not, consider a maximal ideal containing $1 - a_s$ and obtain a contradiction by showing that 1 is in the ideal. Conclude that $x_s \in R\langle x_1, \ldots, x_{s-1} \rangle$, and the claim by induction.)

2. Let $J$ be a two-sided nilpotent ideal of $R$. Show that $J$ is contained in the Jacobson radical.

3. Suppose that $R$ is Artinian. Show that $N = J(R)$ is a nilpotent ideal.

(Hint: Let $N^\infty$ be the ideal to which the chain $N \supset N^2 \supset \ldots$ terminates. Suppose that $N^\infty$ is non-zero and let $I$ be a minimal left ideal in $N^\infty$ such $NI$ is non-zero. Argue that $I$ finitely generated and obtain a contradiction using Nakayama’s lemma.)

4. Let $F$ be a field and $R = M_2(F)$, the ring of $2 \times 2$ matrices over $F$. Exhibit a left ideal in $R$ that is not a semisimple ring.

5. Suppose $F$ is a field and $f(x) \in F[x]$. Set $R = F[x]/(f(x))$. Determine $J(R)$. Under what condition is $R$ semisimple? (Recall the example $\mathbb{Z}/(n)$ discussed in class.)

6. If $F$ is a field, let $R$ be the ring of $n \times n$ upper triangular matrices with entries from $F$. (That is, $A = (a_{ij}) \in R$, then $a_{ij} = 0$ if $i > j$.) Find $J(R)$ and show that $R/J(R)$ is commutative.